Transportation problem
- Formulation of the problem
- Simplex table for TP
- Starting methods
- Unbalanced TP
- Degenerate solution of TP

Assignment Problem
- Hungarian method

Summary

OPERATIONAL ANALYSIS

Basic Course

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Contents

1 **Transportation problem**
- Formulation of the problem
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- Starting methods
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- Degenerate solution of TP

2 **Assignment Problem**
- Hungarian method

3 **Summary**

Transportation problem

Formulation of the transportation problem

- m suppliers $(D_1, D_2, \dots, D_m)$,
- n customers $(O_1, O_2, \dots, O_n)$,
- capacity of suppliers $(a_1, a_2, \dots, a_m)$,
- customer requirements $(b_1, b_2, \dots, b_n)$,
- c_{ij} - the cost of transporting a unit of goods from i-th supliers to the j-th customer,
- x_{ij} - the number of units of goods transported from i-th supliers to the j-th customer.

◆ The goal of solving the traffic problem is to establish such a distribution plan (i. e. x_{ij}) that meets the requirements while maintaining capacities with the lowest possible costs.

Transportation problem

Standard form of TP

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij} \quad & \longrightarrow \quad \min \qquad\qquad (1)\\
\sum_{j=1}^{n} x_{ij} \quad & = \quad a_i \quad \text{pre } i = 1, 2, \ldots, m\\
\sum_{i=1}^{m} x_{ij} \quad & = \quad b_j \quad \text{pre } j = 1, 2, \ldots, n\\
x_{ij} \quad & \geqq \quad 0 \quad \text{pre } i = 1, 2, \ldots, m \text{ a}\\
& \qquad\quad\;\; \text{pre } j = 1, 2, \ldots, n,
\end{aligned}$$

Transportation problem

Balanced TP

Definition: (Balanced TP)

If for a transportation problem (1) applies $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, the given transportation problem is called balanced. Otherwise, we call it unbalanced.

Theorem:

The basic feasible solution of the balanced transportation problem with m suppliers and n customers contains at most $m + n - 1$ non-zero values of x_{ij}.

♦ Every transportation problem can be solved by the simplex method, but it contains too many variables. We write it in a special simplex table intended for TP.

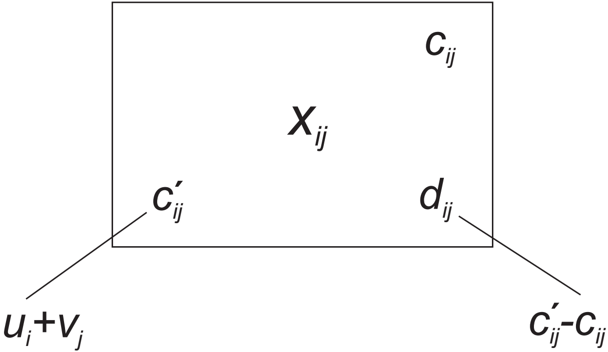
Transportation problem

Simplex Table for TP

—	O_1 v_1	$\dots$	O_n v_n
D_1 u_1	c_{11} x_{11} c'_{11} d_{11}	$\dots$	c_{1n} x_{1n} c'_{1n} d_{1n}
$\vdots$	$\vdots$		$\vdots$
D_m u_m	c_{m1} x_{m1} c'_{m1} d_{m1}	$\dots$	c_{mn} x_{mn} c'_{mn} d_{mn}
b_j	b_1	$\dots$	b_n

Transportation problem

Simplex Table for TP



- u_i and v_j we get from the optimality test,
- $c'_{ij} = u_i + v_j$,
- $d_{ij} = c'_{ij} - c_{ij}$.

Transportation problem

Procedure for TP

1. We write the transportation problem in the modifying simplex table.

Transportation problem

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2. Using one of the starting methods, we can find some basic feasible solution, the so-called starting the task solution.

Transportation problem

Procedure for TP

1. We write the transportation problem in the modifying simplex table.

2. Using one of the starting methods, we can find some basic feasible solution, the so-called starting the task solution.

3. We will test the feasible solution that we found using one of the starting methods or by pivoting to see if it is optimal.
   - if the solution is optimal ⇒ we will finish,
   - if not optimal ⇒ we pivot the table and do the optimality test again, ⇒ we continue with step 3.

Transportation problem

Example of TP – Part 1

The STAVIVA company has an order for bricks for three construction sites S1, S2, S3. The requirements of construction sites are for 600, 400 and 300 pallets of bricks respectively. STAVIVA has 500, 300 and 500 pallets of bricks available in its warehouses V1, V2, V3. The costs of transporting one pallet from the relevant large warehouse to a specific construction site are shown in the following table.

	S1	S2	S3
V1	5	10	8
V2	15	4	11
V3	9	7	6

How should STAVIVA company supply construction sites to keep delivery costs to a minimum? (Write in the modified simplex table.)

Transportation problem

Example of TP – Part 1 (Solution)

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i
V_1 u_1	5	10	8	500
V_2 u_2	15	4	11	300
V_3 u_3	9	7	6	500
b_j	600	400	300	1300

Transportation problem

Initial methods

The starting methods are used to find some (initial) basic feasible solution of the transportation problem (this solution may not be optimal).

Methods:

- the Northwest Corner Method,
- the Index Method,
- the Vogel's Approximation Method,
- the Russell's Method,
- the row (column) minima method,
- etc.

Transportation problem

The Northwest Corner Method

1. Among the basic variables, the variable in the upper left corner (NW) cell is selected and we assign its value $x_{ij} = \min\{a_i, b_j\}$.

Transportation problem

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2. If $x_{ij} = a_i$, then we do not take the i-th row into consideration.

Transportation problem

The Northwest Corner Method

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2. If $x_{ij} = a_i$, then we do not take the i-th row into consideration.
3. If $x_{ij} = b_j$, then we do not take the j-th column into consideration.

Transportation problem

The Northwest Corner Method

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2. If $x_{ij} = a_i$, then we do not take the i-th row into consideration.
3. If $x_{ij} = b_j$, then we do not take the j-th column into consideration.
4. In this way, a (imaginally) smaller table is created and we subtract the value of x_{ij} from the corresponding b_j (or a_i).

Transportation problem

The Northwest Corner Method

1. Among the basic variables, the variable in the upper left corner (NW) cell is selected and we assign its value $x_{ij} = \min\{a_i, b_j\}$.
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4. In this way, a (imaginally) smaller table is created and we subtract the value of x_{ij} from the corresponding b_j (or a_i).
5. In this reduced table, we select the upper left cell again and repeat the previous procedure.

Transportation problem

The Northwest Corner Method

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2. If $x_{ij} = a_i$, then we do not take the i-th row into consideration.
3. If $x_{ij} = b_j$, then we do not take the j-th column into consideration.
4. In this way, a (imaginally) smaller table is created and we subtract the value of x_{ij} from the corresponding b_j (or a_i).
5. In this reduced table, we select the upper left cell again and repeat the previous procedure.
6. We finish if the entire table is filled.

Example:

Using the NW corner method, find a distribution plan for the STAVIVA company from the previous example.

Transportation problem

The Northwest Corner Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i
V_1 u_1	5	10	8	500
V_2 u_2	15	4	11	300
V_3 u_3	9	7	6	500
b_j	600	400	300	1300

Transportation problem

The Northwest Corner Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i
V_1 u_1	5 500	10 —	8 —	500
V_2 u_2	15	4	11	300
V_3 u_3	9	7	6	500
b_j	600	400	300	1300

Transportation problem

The Northwest Corner Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i
V_1 u_1	5 500	10 —	8 —	500
V_2 u_2	15 100	4	11	300
V_3 u_3	9 —	7	6	500
b_j	600	400	300	1300

Transportation problem

The Northwest Corner Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i
V_1 u_1	5 500	10 —	8 —	500
V_2 u_2	15 100	4 200	11 —	300
V_3 u_3	9 —	7	6	500
b_j	600	400	300	1300

Transportation problem

The Northwest Corner Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i
V_1 u_1	5 500	10 —	8 —	500
V_2 u_2	15 100	4 200	11 —	300
V_3 u_3	9 —	7 200	6	500
b_j	600	400	300	1300

Transportation problem

The Northwest Corner Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i
V_1 u_1	5 500	10 —	8 —	500
V_2 u_2	15 100	4 200	11 —	300
V_3 u_3	9 —	7 200	6 300	500
b_j	600	400	300	1300

$$f(\vec{x}) = 5\,.\,500 + 15\,.\,100 + 4\,.\,200 + 7\,.\,200 + 6\,.\,300 = 8000$$

Transportation problem

The Vogel's Approximation Method

1. In each row and column we calculate and write the first differences (I) (i.e. the differences between the smallest and the second smallest price).

Transportation problem

The Vogel's Approximation Method

1. In each row and column we calculate and write the first differences (I) (i.e. the differences between the smallest and the second smallest price).
2. We select the column (row) with the biggest difference. We want to meet the requirements as much as possible (max. we occupy the cells with the lowest price). We omit the filled column (row).

Transportation problem

The Vogel's Approximation Method

① In each row and column we calculate and write the first differences (I) (i.e. the differences between the smallest and the second smallest price).

② We select the column (row) with the biggest difference. We want to meet the requirements as much as possible (max. we occupy the cells with the lowest price). We omit the filled column (row).

③ We recalculate new differences in rows (columns), thereby creating a new column (row) of differences (II).

Transportation problem

The Vogel's Approximation Method

1. In each row and column we calculate and write the first differences (I) (i.e. the differences between the smallest and the second smallest price).
2. We select the column (row) with the biggest difference. We want to meet the requirements as much as possible (max. we occupy the cells with the lowest price). We omit the filled column (row).
3. We recalculate new differences in rows (columns), thereby creating a new column (row) of differences (II).
4. We repeat steps 2 and 3 until we get an acceptable solution (we fill in the entire table).

Transportation problem

The Vogel's Approximation Method

1. If, when calculating the differences, we get the same largest differences in several rows or columns, then we look for the saddle point (the field with the lowest price in terms of rows and columns). We take the row or column that contains it.

Transportation problem

The Vogel's Approximation Method

1. If, when calculating the differences, we get the same largest differences in several rows or columns, then we look for the saddle point (the field with the lowest price in terms of rows and columns). We take the row or column that contains it.
2. If during the calculation we have several saddle points at once, then we decide on the one with the lowest sum of indexes indicating the respective row and column (lexicographic rule).

Transportation problem

The Vogel's Approximation Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i	I
V_1 u_1	5	10	8	500	3
V_2 u_2	15	4	11	300	7
V_3 u_3	9	7	6	500	1
b_j	600	400	300	1300	
I	4	3	2		

Transportation problem

The Vogel's Approximation Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i	I
V_1 u_1	5	10	8	500	3
V_2 u_2	15	4 300	11	300	7
V_3 u_3	9	7	6	500	1
b_j	600	400	300	1300	
I	4	3	2		

Transportation problem

The Vogel's Approximation Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i	I
V_1 u_1	5	10	8	500	3
V_2 u_2	—	300	—	300	x
V_3 u_3	9	7	6	500	1
b_j	600	400	300	1300	
I	4	3	2		
II	4	3	2		

Transportation problem

The Vogel's Approximation Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i	I
V_1 u_1	5 500	10	8	500	3
V_2 u_2	—	300	—	300	x
V_3 u_3	9	7	6	500	1
b_j	600	400	300	1300	
I	4	3	2		
II	4	3	2		

Transportation problem

The Vogel's Approximation Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i	I
V_1 u_1	500	—	—	500	x
V_2 u_2	—	300	—	300	x
V_3 u_3	9 100	7 100	6 300	500	1
b_j	600	400	300	1300	
I	4	3	2		
II	4	3	2		

Transportation problem

The Vogel's Approximation Method – Example

—	S_1 v_1	S_2 v_2	S_3 v_3	a_i	I
V_1 u_1	500	—	—	500	x
V_2 u_2	—	300	—	300	x
V_3 u_3	9 100	7 100	6 300	500	1
b_j	600	400	300	1300	
I	4	3	2		
II	4	3	2		

$$f(\vec{x}) = 5\,.\,500 + 4\,.\,300 + 9\,.\,100 + 7\,.\,100 + 6\,.\,300 = 7100$$

Transportation problem

Optimality Test

1. We construct an equation for each basic cell $u_i + v_j = c_{ij}$.

Transportation problem

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2. We assign any value to one of the variables (parameters) (e.g. $v_3 = 0$) and calculate the others with respect to the parameter.

Transportation problem

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1. We construct an equation for each basic cell $u_i + v_j = c_{ij}$.
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3. We write the calculated values of u_i and v_j in the table.

Transportation problem

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3. We write the calculated values of u_i and v_j in the table.
4. We calculate the values $c'_{ij} = u_i + v_j$ of each cell and write them in the table.

Transportation problem

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2. We assign any value to one of the variables (parameters) (e.g. $v_3 = 0$) and calculate the others with respect to the parameter.
3. We write the calculated values of u_i and v_j in the table.
4. We calculate the values $c'_{ij} = u_i + v_j$ of each cell and write them in the table.
5. We calculate $d_{ij} = c'_{ij} - c_{ij}$ and write all values in the table (for basic cells it must be $c'_{ij} = c_{ij}$).

Transportation problem

Optimality Test

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3. We write the calculated values of u_i and v_j in the table.
4. We calculate the values $c'_{ij} = u_i + v_j$ of each cell and write them in the table.
5. We calculate $d_{ij} = c'_{ij} - c_{ij}$ and write all values in the table (for basic cells it must be $c'_{ij} = c_{ij}$).
6. If all values of $d_{ij} \leq 0$, then this basic feasible solution is optimal.

If the given basic feasible solution is not optimal, we pivot the table, then after pivoting we do the optimality test again.

Transportation problem

Optimality Test – Example

Test whether the distribution plans for STAVIVA company obtained using the NW corner method are optimal.

—	S_1 v_1	S_2 v_2	S_3 v_3
V_1 u_1	5 500	10 —	8 —
V_2 u_2	15 100	4 200	11 —
V_3 u_3	9 —	7 200	6 300

Transportation problem

Optimality Test – Example

For each basic cell, we construct the equation $u_i + v_j = c_{ij}$:

$$u_1 + v_1 = 5$$
$$u_2 + v_1 = 15$$
$$u_2 + v_2 = 4$$
$$u_3 + v_2 = 7$$
$$u_3 + v_3 = 6$$

Transportation problem

Optimality Test – Example

For each basic cell, we construct the equation $u_i + v_j = c_{ij}$:

$$
\begin{aligned}
u_1 + v_1 &= 5\\
u_2 + v_1 &= 15\\
u_2 + v_2 &= 4\\
u_3 + v_2 &= 7\\
u_3 + v_3 &= 6
\end{aligned}
$$

Let $u_1 = 0$

Transportation problem

Optimality Test – Example

For each basic cell, we construct the equation $u_i + v_j = c_{ij}$:

$$u_1 + v_1 = 5$$

$$u_2 + v_1 = 15$$

$$u_2 + v_2 = 4$$

$$u_3 + v_2 = 7$$

$$u_3 + v_3 = 6$$

Let $u_1 = 0$ $\quad v_1 = 5$

$$u_2 = 10$$

$$v_2 = -6$$

$$u_3 = 13$$

$$v_3 = -7$$

Transportation problem

Optimality Test – Example

—	S_1 5	S_2 −6	S_3 −7
V_1 0	5 500	10 —	8 —
V_2 10	15 100	4 200	11 —
V_3 13	9 —	7 200	6 300

Transportation problem

Optimality Test – Example

Let's calculate the values $c'_{ij} = u_i + v_j$ of each cell:

—	S_1 5	S_2 -6	S_3 -7
V_1 0	5 500 5	10 — -6	8 — -7
V_2 10	15 100 15	4 200 4	11 — 3
V_3 13	9 — 18	7 200 7	6 300 6

Transportation problem

Optimality Test – Example

Let's calculate the values $d_{ij} = c'_{ij} - c_{ij}$:

—	S_1 5	S_2 -6	S_3 -7
V_1 0	5 500 5 0	10 — -6 -16	8 — -7 -15
V_2 10	15 100 15 0	4 200 4 0	11 — 3 -8
V_3 13	9 — 18 9	7 200 7 0	6 300 6 0

Transportation problem

Optimality Test – Example

Let's calculate the values $d_{ij} = c'_{ij} - c_{ij}$:

—	S_1 5	S_2 -6	S_3 -7
V_1 0	5 500 5 0	10 — -6 -16	8 — -7 -15
V_2 10	15 100 15 0	4 200 4 0	11 — 3 -8
V_3 13	9 — 18 9	7 200 7 0	6 300 6 0

The value $d_{31} > 0$ and this basic feasible solution is not optimal, therefore we pivot the table.

Transportation problem

Table Pivoting

1. In the empty cell with the largest value d_{ij}, we add a sign in the upper left corner $\oplus$.

Transportation problem

Table Pivoting

1. In the empty cell with the largest value d_{ij}, we add a sign in the upper left corner $\oplus$.
2. We create a cycle of alternating signs by starting in the marked cell $\oplus$ and in the table we can:
   - move only up, down, left and right,
   - we can change the direction only on an occupied cell.

Transportation problem

Table Pivoting

1. In the empty cell with the largest value d_{ij}, we add a sign in the upper left corner $\oplus$.
2. We create a cycle of alternating signs by starting in the marked cell $\oplus$ and in the table we can:
   - move only up, down, left and right,
   - we can change the direction only on an occupied cell.

   We gradually add $\ominus$ or $\oplus$ to the upper left corner of these occupied cells, in which we have changed direction, until we return to the cell from which we started.

Transportation problem

Table Pivoting

1. In the empty cell with the largest value d_{ij}, we add a sign in the upper left corner $\oplus$.
2. We create a cycle of alternating signs by starting in the marked cell $\oplus$ and in the table we can:
   - move only up, down, left and right,
   - we can change the direction only on an occupied cell.

   We gradually add $\ominus$ or $\oplus$ to the upper left corner of these occupied cells, in which we have changed direction, until we return to the cell from which we started.
3. We take the minimum from the minus cells, then we distribute it over the cycle according to the signs, i.e. we either add or subtract this minimum to the given x_{ij} values, which gives us a new basic feasible solution.

Transportation problem

Table Pivoting – Example

Example:
Find the optimal delivery plan for STAVIVA company.

—	S_1 5	S_2 -6	S_3 -7
V_1 0	5 500 5 0	10 — -6 -16	8 — -7 -15
V_2 10	⊖ 15 100 15 0	⊕ 4 200 4 0	11 — 3 -8
V_3 13	⊕ 9 — 18 9	⊖ 7 200 7 0	6 300 6 0

Transportation problem

Table Pivoting – Example

We test optimality again.

—	S_1 v_1	S_2 v_2	S_3 v_3
V_1 u_1	5 500	10 —	8 —
V_2 u_2	⊖ 15 —	⊕ 4 300	11 —
V_3 u_3	⊕ 9 100	⊖ 7 100	6 300

Transportation problem

Table Pivoting – Example

We construct an equation for each basic cell $u_i + v_j = c_{ij}$:

$$\begin{aligned} u_1 + v_1 &= 5 \\ u_2 + v_2 &= 4 \\ u_3 + v_1 &= 9 \\ u_3 + v_2 &= 7 \\ u_3 + v_3 &= 6 \end{aligned}$$

Transportation problem

Table Pivoting – Example

We construct an equation for each basic cell $u_i + v_j = c_{ij}$:

$$u_1 + v_1 = 5$$
$$u_2 + v_2 = 4$$
$$u_3 + v_1 = 9$$
$$u_3 + v_2 = 7$$
$$u_3 + v_3 = 6$$

Let $u_1 = 0$

Transportation problem

Table Pivoting – Example

We construct an equation for each basic cell $u_i + v_j = c_{ij}$:

$$u_1 + v_1 = 5$$
$$u_2 + v_2 = 4$$
$$u_3 + v_1 = 9$$
$$u_3 + v_2 = 7$$
$$u_3 + v_3 = 6$$

Let $u_1 = 0$ $\quad v_1 = 5$

$$u_3 = 4$$
$$v_2 = 3$$
$$u_2 = 1$$
$$v_3 = 2$$

Transportation problem

Table Pivoting – Example

—	S_1 5	S_2 3	S_3 2
V_1 0	5 500 5 0	10 — 3 −7	8 — 2 −6
V_2 1	⊖ 15 — 6 −9	⊕ 4 300 4 0	11 — 3 −8
V_3 4	⊕ 9 100 9 0	⊖ 7 100 7 0	6 300 6 0

Transportation problem

Table Pivoting – Example

—	S_1 5	S_2 3	S_3 2
V_1 0	5 500 5 0	10 — 3 −7	8 — 2 −6
V_2 1	⊖ 15 — 6 −9	⊕ 4 300 4 0	11 — 3 −8
V_3 4	⊕ 9 100 9 0	⊖ 7 100 7 0	6 300 6 0

All values of $d_{ij} \leq 0$, therefore this BFS is optimal:
$f^{opt} = 5\,.\,500 + 4\,.\,300 + 9\,.\,100 + 7\,.\,100 + 6\,.\,300 = 7100.$

Transportation problem

An Unbalanced Transportation Problem

1. $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j \Rightarrow$ the total demands of customers are greater than the total capacities of suppliers, therefore we add a dummy (virtual, phantom) supplier D_{m+1} with capacity

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$$

and all prices $c_{m+1,j} = 0$, for $j = 1, 2, \ldots, n$.

Transportation problem

An Unbalanced Transportation Problem

1. $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j \Rightarrow$ the total demands of customers are greater than the total capacities of suppliers, therefore we add a dummy (virtual, phantom) supplier D_{m+1} with capacity
$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$$
and all prices $c_{m+1,j} = 0$, for $j = 1, 2, \dots, n$.

2. $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j \Rightarrow$ the total capacities of suppliers exceed the total demands of customers, so we add a dummy customer O_{n+1} with a request
$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$$
and all prices $c_{i,n+1} = 0$, for $i = 1, 2, \dots, m$.

Transportation problem

An Unbalanced Transportation Problem – Example

Enterprise MLYNY, s.r.o. from its three mills (M_1, M_2, M_3) supplies flour to 4 bakeries (P_1, P_2, P_3, P_4). The costs of transporting one ton of flour from mills to bakeries as well as their requirements and capacities are shown in the table. Create a flour delivery plan for the lowest cost.

	P_1	P_2	P_3	P_4	a_i
M_1	12	9	7	13	380
M_2	6	15	10	11	400
M_3	7	14	17	9	350
b_j	330	280	300	250	$1160 \setminus 1130$

Transportation problem

Degenerate solution

Definition: (Degenerate solution)

If the number of nonzero values for x_{ij} (number of occupied (filled out) cells) is less than $m + n - 1$, then the solution is called **degenerate**.

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- when calculating the feasible solution (if in the input for some i, j applies $a_i = b_j$)
- during pivoting (if there are the same value in several fields with $\ominus$ – the minimum one)

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Degeneration removal:

$\Rightarrow$ fill the fill out cells with zeros so that there are just $m + n - 1$ fill outcells (We have to choose so that all blocks are connected, so they don't just touch the corners)

Transportation problem

Degenerate solution – Example

Find the optimal solution for the transport problem given in the following table:

	O_1 v_1	O_2 v_2	O_3 v_3	a_i
D_1 u_1	20	40	50	8
D_2 u_2	40	40	50	14
D_3 u_3	50	20	70	6
b_j	8	9	11	28

Assignment Problem

Assignment problem formulation

- Another type of linear programming problem can be said to be a subtype of a transportation problem
- it is a matter of assigning a certain number of objects to the same number of destinations so that the objective function is minimal resp. maximal
- the objective function can be a function of distance, time, cost, efficiency,. . .
- it depends on the specific case whether the task will be minimizing or maximizing.

Assignment Problem

Assignment problem formulation

- n objects $(O_1, O_2, \ldots, O_n)$
- n destinations $(M_1, M_2, \ldots, M_n)$
- c_{ij} – rates (prices) for the relationship between i^{th} object and j^{th} destination
- x_{ij} – variable expressing whether i^{th} object will be assigned to j^{th} destination

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Differences compared to the transportation problem:

- the problem is integer
- all "supplier capacities" and "customer requirements" are equal 1
- x_{ij} – a variable takes on the values 0 or 1 only

Assignment Problem

Standard form of the assignment problem

$$f(\boldsymbol{x}) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij}\, x_{ij}) \to \max$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{pre } i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \text{pre } j = 1, 2, \ldots, n$$

$$x_{ij} \in \{0, 1\} \qquad \text{pre } i, j = 1, 2, \ldots, n$$

Assignment Problem – Hungarian method

Hungarian method – Description

- ⇒ the assignment problem is considerably degenerate - it contains only n non-zero values so it is maximally degenerate
- ⇒ the result is a large amount of inefficient pivoting
- ⇒ for solving of the Assignment Problem is used the so-called **Hungarian method** (but the objective function must be minimized)

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Hungarian method – Description

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⇒ the result is a large amount of inefficient pivoting

⇒ for solving of the Assignment Problem is used the so-called **Hungarian method** (but the objective function must be minimized)

Hungarian Method Algorithm:

1. Reduction of the rate matrix (square) ⇒ obtain the initial solution.
2. Finding Independent Zeros ⇒ Finding the optimal solution if we have n independent zeros.
3. Construction of covering lines.
4. Further reduction of the rate matrix.
5. Return to point 2, or find the optimal solution.

Assignment Problem – Hungarian method

Hungarian method – 1. Rate matrix reduction

1. There is a minimum rate of c_{ij} in each line, which is subtracted from the other rates in the line $\Rightarrow$ to ensure that at least one zero reduced rate is created in each line.

Assignment Problem – Hungarian method

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Assignment Problem – Hungarian method

Hungarian method – 1. Rate matrix reduction

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2. It is checked whether there are zero reduced rates in each column as well.
3. If not, a rate reduction will also be performed in those columns where there has been no zero rate so far, thus ensuring that there is at least one zero reduced rate in each row and column.

Assignment Problem – Hungarian method

Hungarian method – 2. Finding independent zeros

We will try to assign the maximum possible number of non-zero values to variables with zero reduced rates, preferentially assigning non-zero values to variables in which rows or columns there is a minimum number of zero rates (preferably only one)

⇒ if all objects can be assigned to the destination in this way (i.e. if the number of all non-zero values of the variables x_{ij} is equal to the dimension of the task n) ⇒ we have an optimal solution and the value of its objective function must be determined from the original rate matrix,

⇒ if not, we need to make adjustments to the ⇒ cover lines construction.

Assignment Problem – Hungarian method

Hungarian method – 3. Construction of cover lines

1. All variables with zero reduced rates cover the minimum number of so-called cover lines. At the same time, firstly are covered these rows or columns in which there are maximum number of zero reduced rates. According to the so-called a König theorem, the number of cover lines should be equal to the number obtained non-zero variables (at the same time it is also a check of the correctness of the solution).

Assignment Problem – Hungarian method

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2. The minimum rate for uncovered variables is found and it is:
   - **I.** reduced from the rates of the non-covering variables.
   - **II.** added to the rates of the twice-covered variables (where the cover lines intersect).
   - **III.** The rates of the variables that are covered once remain unchanged.

Assignment Problem – Hungarian method

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⇒ this procedure creates a new matrix of reduced rates and repeats the process

Assignment Problem – Hungarian method

Hungarian method – Example

Company STAVMAT has 3 cranes at its disposal, which it needs to move to 3 constructions so that the costs are minimal. The costs of transporting specific cranes to specific constructions are listed in the following table.

	S1	S2	S3
Z1	4	3	1
Z2	1	2	6
Z3	4	5	3

Linear Optimization

Summary

THANK YOU FOR YOUR ATTENTION.