TECHNICKÁ UNIVERZITA V KOŠICIACH - VÝZVA pre tvoju BUDÚCNOSŤ

Moderné vzdelávanie pre vedomostnú spoločnosť/
Projekt je spolufinancovaný zo zdrojov EÚ

Mathematics 3

University Textbook

Fakulta elektrotechniky a informatiky

Štefan Berežný and Daniela Kravecová

Európska únia
Európsky sociálny fond

Agentúra Ministerstva školstva, vedy, výskumu a športu SR pre štrukturálne fondy EÚ

Operačný program VZDELÁVANIE

Technical University of Košice
Faculty of Electrical Engineering and Informatics
Department of Mathematics and Theoretical Informatics

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Košice 2014

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First Edition

Táto publikácia vznikla za finančnej podpory

z **Európskeho sociálneho fondu** v rámci
Operačného programu **VZDELÁVANIE**.

Prioritná os 1 Reforma vzdelávania a odbornej prípravy
Opatrenie 1.2 Vysoké školy a výskum a vývoj ako motory
rozvoja vedomostnej spoločnosti.

Názov projektu: Balík prvkov pre skvalitnenie a inováciu vzdelávania na TUKE
ITMS 2611020070

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Editor: Technical University of Košice
Faculty of Electrical Engineering and Informatics

ISBN: 978-80-553-1790-8

Preface

The textbook Mathematics 3 contains an overview of the theory, solved examples and unsolved tasks for subject Mathematics 3 for bachelor's degrees students at Applied Informatics, Faculty of Electrical Engineering and Informatics, Technical University of Košice. The textbook consists of six chapters. Each chapter is divided into sub-chapters in particular areas of Mathematics. At the end of each chapter are subsections Solved Examples, Exercises and their results.

The areas of the mathematics optimization are represented in this textbook by the theory, examples, and basic information from operational analysis and simplex method, as this required course of study Applied Informatics.

This textbook is available on CD and on the web site DMTI FEEI TUKE (KMTI FEI TU) and Moodle system, which is managed by the FEEI TUKE.

Košice, 31st of August 2014 Authors

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List of Abbreviations and Symbols

MPP – mathematical programming problem

LPP – linear programming problem

$\boldsymbol{a_i}$ – i-th row of the matrix $\boldsymbol{A}$

A_j – j-th column of the matrix $\boldsymbol{A}$

F – set of feasible solutions of LPP

$\boldsymbol{x}^{opt}$ – optimal solution of LPP

$f^{opt}(\boldsymbol{x})$ – value of the objective function at the optimal solution

$conv(M)$ – convex hull of the set M

$ex(M)$ – set of extreme points of the set M

$B(i)$ – index of the matrix $\boldsymbol{A}$ column which represent i-th component of base $\mathcal{B}$

BS – basis solution

P – primary linear programming problem

D – dual linear programming problem

F_P – set of feasible solutions of primary linear programming problem

F_D – set of feasible solutions of dual linear programming problem

ILPP – integer linear programming problem

$\boldsymbol{x}_r^{opt}$ – optimal solution of relaxation of integer linear programming problem

$\overline{BC}$ – line segment BC

$\{a\}$ – fractional part of a

$\lfloor a \rfloor$ – lower integer part of a

TP – transportation problem

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Chapter 1

Introduction to Linear Programming

1.1 Historical Introduction

Linear programming is a relatively young mathematical discipline, dating from the invention of the simplex method by G. B. Dantzig in 1947. Historically, development in linear programming is driven by its applications in economics and management. Dantzig initially developed the simplex method to solve U.S. Air Force planning problems, and planning and scheduling problems still dominate the applications of linear programming. One reason that linear programming is a relatively new field is that only the smallest linear programming problems can be solved without a computer.

1.2 Mathematical Programming

Methods of mathematical programming are some of the frequently used methods to optimize production and other decision-making processes. They allow to transform realistic processes to mathematical models and then to solve these models by using the mathematical tools. So the real process is transformed into *mathematical programming problem* – MPP.

Parts of the mathematical programming problem are:

1. objectives - determining of objectives is dependent on the process. They are the optimization (maximization or minimization) criteria. These can be, for example:

   – profit maximization,

   – maximization of efficiency equipment,

   – maximization productivity,

   – maximization the quantities of material,

   – minimization of production costs,

   – waste minimization,

  - minimization of kilometers and other.

2. Constraints - they refer to conditions and limitations so the process is working. It may be one but also a number of conditions. For example:

   - material resources,
   - capacity of the production facilities,
   - workforce capacity,
   - limited lifetime of machines,
   - financial resources,
   - sales opportunities,
   - suppliers capacity ,
   - transport capacity
   - requirements of customers,
   - storage capacity and other.

Objectives and conditions can be expressed by using mathematical tools, which we call a *mathematical model*.

Objectives are functions, which we are trying to minimize or maximize.

Constraints are given by equalities, inequalities or by system of equalities and inequalities. They may also be linear or nonlinear equalities and inequalities.

Table 1.1: Optimization problem.

objectives (objective functions)	$f_1(x_1, x_2, \ldots, x_n) \to \min(\max)$ $f_2(x_1, x_2, \ldots, x_n) \to \min(\max)$ $\ldots$ $f_k(x_1, x_2, \ldots, x_n) \to \min(\max)$
main constraints	$g_1(x_1, x_2, \ldots, x_n) \le\ge= 0$ $g_2(x_1, x_2, \ldots, x_n) \le\ge= 0$ $\ldots$ $g_m(x_1, x_2, \ldots, x_n) \le\ge= 0$

The mathematical model of MPP is illustrated by the following example.

Example 1.1. A carpenter makes 2 products A and B. Each piece of A can be sold for a profit of 65 € and each piece of B for a profit of 48 €. The carpenter can afford to spend up to 90 hours per week working and takes four hours to make A and nine hours to make B. The final treatment of products takes two hours for A and one hour for B and the carpenter can afford to spend up to 20 hours per week. Each piece of the product occupies 1 m^3 in storage and capacity of storage place is 12 m^3. Formulate this problem as a *linear programming problem.*
Solution:
We enter the relevant data in the following table:

Table 1.2: Table of requirements and capabilities.

	product A	product B	capacities
working time	4	9	90
final treatment time	2	1	20
storace place	1	1	12
profit	65	48	–

The process can be expressed by the following mathematical model

$$f(x_1, x_2) = 65x_1 + 48x_2 \rightarrow \max$$

$$4x_1 + 9x_2 \leq 90$$

$$2x_1 + x_2 \leq 20$$

$$x_1 + x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

$\surd$

Chapter 2

Linear Programing Problem

2.1 Basic Concepts

Definition 2.1. A function $f(\boldsymbol{x})$ of several real variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is said to be linear if it satisfies two conditions:

(1) $f(\boldsymbol{x} + \boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y})$ additivity,

(2) $f(\alpha \boldsymbol{x}) = \alpha f(\boldsymbol{x})$ proportionality,

where $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Corollary 2.1. All linear functions are on the form

$$f(\boldsymbol{x}) = \sum_{j=1}^{n} (c_j \cdot x_j), \qquad \text{where} \quad c_j \in \mathbb{R}, \quad \forall j \in \{1, 2, \ldots, n\}.$$

Definition 2.2. A linear programming problem is a problem of maximizing or minimizing a linear objective function of n real variables

$$f(\boldsymbol{x}) = c_1 \cdot x_1 + c_2 \cdot x_2 + \cdots + c_n \cdot x_n \to \min \text{ (max)}, \tag{2.1}$$

whose values are restricted (or constrained) to satisfy relations each of which is of the type:

$$\begin{aligned}
a_{i1} \cdot x_1 + a_{i2} \cdot x_2 + \cdots + a_{in} \cdot x_n &\le b_i, \quad \text{for } i = 1, \ldots, k-1 \\
a_{i1} \cdot x_1 + a_{i2} \cdot x_2 + \cdots + a_{in} \cdot x_n &\ge b_i, \quad \text{for } i = k, \ldots, l-1 \\
a_{i1} \cdot x_1 + a_{i2} \cdot x_2 + \cdots + a_{in} \cdot x_n &= b_i, \quad \text{for } i = l, \ldots, m \\
x_i \le 0 &\quad \forall i \in N_1 \\
x_i \ge 0 &\quad \forall i \in N_2 \\
x_i \in \mathbb{R} &\quad \forall i \in N_3 \qquad x_i \text{ is unbounded variable, where} \\
1 \le k \le l &\le m, \\
N_1 \cup N_2 \cup N_3 &= \{1, 2, \ldots, n\}
\end{aligned} \tag{2.2}$$

$x_1, x_2, \ldots, x_n$ – variables,
$c_1, c_2, \ldots, c_n$ – the objective function coefficients,
$a_{11}, a_{12}, \ldots, a_{mn}$ - the coefficients of constraints,
$b_1, b_2, \ldots, b_m$ – the coefficients of right sides.

By using the Corollary 2.1 the LPP can be given by:

$$f(\boldsymbol{x}) = \sum_{j=1}^{n} (c_j \cdot x_j) \to \min\ (\max),$$

under conditions:

$$\sum_{j=1}^{n} (a_{ij} \cdot x_j) \leq b_i, \quad \text{for } i = 1, \ldots, k-1$$

$$\sum_{j=1}^{n} (a_{ij} \cdot x_j) \geq b_i, \quad \text{for } i = k, \ldots, l-1$$

$$\sum_{j=1}^{n} (a_{ij} \cdot x_j) = b_i, \quad \text{for } i = l, \ldots, m$$

$$x_j \leq 0, \quad \forall j \in N_1$$

$$x_j \geq 0, \quad \forall j \in N_2$$

$$x_j \in \mathbb{R}, \quad \forall j \in N_3.$$

The simplified symbolic notation:

$$f(\boldsymbol{x}) = \sum_{j=1}^{n} (c_j \cdot x_j) \to \min\ (\max)$$

$$\sum_{j=1}^{n} (a_{ij} \cdot x_j) \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b_i \qquad i = 1, 2, \ldots, m$$

$$x_j \leq\geq 0 \qquad j = 1, 2, \ldots, n.$$

The vector notation:

$$f(\boldsymbol{x}) = \boldsymbol{c}^\top \cdot \boldsymbol{x} \to \min\ (\max)$$

$$\sum_{i=1}^{m} (\boldsymbol{a_i} \cdot \boldsymbol{x}) \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} \boldsymbol{b}$$

$$x_j \leq\geq 0, \qquad j = 1, 2, \ldots, n \text{ , where}$$

$\boldsymbol{x} = (x_1, x_2, \dots, x_n)^\top$
$\boldsymbol{c} = (c_1, c_2, \dots, c_n)^\top$
$\boldsymbol{a_i} = (a_{i1}, a_{i2}, \dots, a_{in})$
$\boldsymbol{b} = (b_1, b_2, \dots, b_m)^\top$.

The matrix notation:

$$f(\boldsymbol{x}) = \boldsymbol{c}^\top \cdot \boldsymbol{x} \to \min\ (\max)$$

$$\boldsymbol{A} \cdot \boldsymbol{x} \left\{ \begin{matrix} \leq \\ = \\ \geq \end{matrix} \right\} \boldsymbol{b}$$

$$x_j \leq\geq 0, \qquad j = 1, 2, \dots, n,$$

where the matrix $\boldsymbol{A} \in \mathbb{R}_{m,n}$ is the matrix of real numbers with m rows and n columns:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \tag{2.3}$$

Remark 2.1 (Denote)**.** i-th row of the matrix $\boldsymbol{A}$ by $\boldsymbol{a_i}$ and j-th column of the matrix $\boldsymbol{A}$ by A_j.

Definition 2.3.

- A vector $\boldsymbol{x} \in \mathbb{R}^n$ for the LPP is said to be feasible if it satisfies the corresponding constraints.

- The set of all feasible vectors is called the constraint set F.

- A linear programming problem is said to be feasible if the constraint set is not empty; otherwise it is said to be infeasible.

- A feasible vector $\boldsymbol{x} \in \mathbb{R}^n$, at which the objective function (2.1) achieves extremal (maximum or minimum) value is called optimal $\boldsymbol{x}^{opt}$. This extremal value of feasible function is denoted $f^{opt}(\boldsymbol{x})$.

- A feasible LP problem is said to be unbounded if the objective function can assume arbitrarily large positive (resp. negative) values at feasible vectors; otherwise, it is said to be bounded.

2.2 Selected Types of Linear Programming Problems

The linear programming problems offer a large variety of applications in practice. The following structure describes only some of them and examples, which are provided to them, illustrate very simplified procedures.

2.2.1 The Activity Analysis Problem

Linear programming problems arise naturally in production planning. There are n products that a company may product, using the available supplies of m resources (labour, finance, hours, steel, etc.). The company knows the amount of $i-$th resource which is needed to produce a unit of $j-$th product. It is also known, what is the profit from the sale of a unit quantity of each product and the available supply of resources are known too.

The task is to schedule production plan so that the profit will be maximum with respect the capacity of the resources.

Mathematical model:

$$
\begin{aligned}
f(\boldsymbol{x}) = \sum_{j=1}^{n}(c_j \cdot x_j) &\to \max \\
\sum_{j=1}^{n}(a_{ij} \cdot x_j) \leq b_i, &\qquad i = 1, 2, \ldots, m \\
x_j \geq 0, &\qquad j = 1, 2, \ldots, n
\end{aligned}
$$

that

n - number of products,

m - number of resources,

x_j - amount of produced units of the $j-$th product,

c_j - price/profit from a unit quantity of $j-$th product,

a_{ij} - amount of i-th resource used in production of a unit of $j-$th product,

b_i - available supply of i-th resource.

See examples 2.5 and 2.6.

2.2.2 The Diet Problem

There are numbers of different types of food, $F_1, \ldots, F_n$, that supply varying quantities of the nutrients, $N_1, \ldots, N_m$, that are essential to be a good diet. Let us know the minimum (maximum) daily requirement of $i-$th nutrient, the price per unit of $j-$th food and the amount of $i-$th nutrient contained in one unit of $j-$th food. The problem is to supply the required nutrients at minimum cost.

The general mathematical formulation of the problem can be written as follows:

$$
\begin{aligned}
f(\boldsymbol{x}) = \sum_{j=1}^{n}(c_j \cdot x_j) &\to \min \\
\sum_{j=1}^{n}(a_{ij} \cdot x_j) \geq b_i, &\qquad i = 1, 2, \ldots, m \\
x_j \geq 0, &\qquad j = 1, 2, \ldots, n
\end{aligned}
$$

for

m - number of the nutrients,

n - number of different types of food,

x_j - amount of $j-$th food used in the diet,

c_j - price per unit of $j-$th food,

a_{ij} - amount of $i-$th nutrient contained in one unit of $j-$th food,

b_i - minimum (maximum) amount of $i-$th nutrient which is required.

See example 2.7.

2.2.3 The Cutting Plans

We have a certain amount of bars of a given length. We need to cut fixed quantities of a required shorter lengths of them. The target is to establish such a cutting plan – a way in which the bars are to be cut (setting of cutting blades) - to ensure the required amount of bars with required length and waste to a minimum. Waste should be minimized.

The mathematical formulation of the problem can be written as follows:

$$
\begin{aligned}
f(\boldsymbol{x}) = \sum_{j=1}^{n}(c_j \cdot x_j) &\to \min \\
\sum_{j=1}^{n}(a_{ij} \cdot x_j) \geq b_i, &\qquad i = 1, 2, \dots, m \\
x_j \geq 0, &\qquad j = 1, 2, \dots, n
\end{aligned}
$$

for

n - number of different ways of cutting bars (the number of setting options for cutting blades),

m - number of different lengths of bars, we want to cut,

x_j - number of pieces of original bars which are cut by j-th way,

c_j - waste arising from cutting one bar by j-th way,

a_{ij} - number of bars with i-th length cut in j-cutting plan,

b_i - required number of bars of i-th length.

See example 2.8.

2.2.4 The Transportation Problem

There are n providers (companies, contractors, ports,...) that supply a certain commodity and m customers (consumers, markets, clients,...) to which this commodity is taken. Each provider has a certain amount of commodity - capacity and each customer has a specific requirement for the quantity of commodity. The cost of transporting a unit of the commodity are known for each pair of provider - customer. The task is to establish a plan of transportation that costs of which will be as small as possible. (We assume a balanced system, i. e. requirements of customers will be the same as the capacity of providers.)

The general mathematical formulation of the problem can be written as follows:

$$
\begin{aligned}
f(\boldsymbol{x}) = \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij} \cdot x_{ij}) &\to \min \\
\sum_{j=1}^{n} x_{ij} = a_i, &\qquad i = 1, 2, \dots, m \\
\sum_{i=1}^{m} x_{ij} = b_j, &\qquad j = 1, 2, \dots, n \\
x_{ij} \geq 0, &\qquad \text{pre } i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n
\end{aligned}
$$

that

n - number of providers,

m - number of customers,

x_{ij} - number of units of commodity to be transported from the i-th provider to the j-th customer,

c_{ij} - cost of transporting of the commodity from the i-th provider to the j-th customer,

a_i - capacity of the i-th provider,

b_j - requirement of the j-th customer.

The balancing condition:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

See example 2.9.

2.2.5 The Assignment Problem

The assignment problem is one of the special cases of transportation problems. The goal of the assignment problem is to minimize the cost or time of completing a number of sources by a number of destinations. An important characteristic of the assignment problem is the one in which the number of sources is equal to the number of destinations.

We have n sources (people, machines, laborers,...) and n destinations (jobs, places, tasks,...) to be assigned to n sources. No source can either be idle or be assigned to more than one destination. Every pair of "source – destination" has a rating expressed by the coefficient c_{ij}. This rating may be cost, satisfaction, penalty involved or time taken to finish the job. Thus, the assignment problem is to find such "source - destination" combinations that optimize the sum of ratings among all. Variables will state whether the source is assigned to a given destination or not.
Thus:

$$x_{ij} = \begin{cases} 1, & \text{if the } i\text{-th source is assigned to the } j\text{-th destination,} \\ 0, & \text{if the } i\text{-th source is not assigned to the } j\text{-th destination.} \end{cases}$$

Since the assignment problem is one of the special cases or modification of the transportation problem, the general mathematical formulation of the assignment problem is similar to the mathematical model of the transportation problem:

$$
\begin{aligned}
f(\boldsymbol{x}) = \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij} \cdot x_{ij}) &\to \max \\
\sum_{j=1}^{n} x_{ij} = 1, \qquad & i = 1, 2, \ldots, n \\
\sum_{i=1}^{n} x_{ij} = 1, \qquad & j = 1, 2, \ldots, n \\
x_{ij} \in \{0, 1\}, \quad & \text{for } i, j = 1, 2, \ldots, n
\end{aligned}
$$

that

- n – number of sources and also the number of destinations,
- x_{ij} – variable that indicates whether the i-th source is assigned to the j-th destination or not,
- c_{ij} – coefficient, which expresses rating of the pair "i-th source - j-th destination".

See example 2.10.

2.3 Linear Programming Problem in $\mathbb{R}^2$

The general mathematical formulation of the linear problem with two decision variables can be written as following:

$$
\begin{aligned}
f(\boldsymbol{x}) = c_1 \cdot x_1 + c_2 \cdot x_2 &\to \min\ (\max) \\
a_{i1} \cdot x_1 + a_{i2} \cdot x_2 \leq b_i, &\quad \text{for } i = 1, \ldots, k-1 \\
a_{i1} \cdot x_1 + a_{i2} \cdot x_2 \geq b_i, &\quad \text{for } i = k, \ldots, l-1 \\
a_{i1} \cdot x_1 + a_{i2} \cdot x_2 = b_i, &\quad \text{for } i = l, \ldots, m \\
x_1, x_2 &\leq\geq 0, \\
1 \leq k &\leq l \leq m
\end{aligned}
$$

Since all constraints are given by linear equations or inequalities, so they can be illustrated by lines or half-planes in the plane. The set of feasible solutions F will be the intersection of these lines and half-planes.

Example 2.1. Let the linear programming problem be given by the following:

$$
\begin{aligned}
f(\boldsymbol{x}) = 66x_1 + 48x_2 &\to \max \\
4x_1 + 9x_2 \leq 90 &\quad \ldots \text{it corresponds to plain } p_1 \\
2x_1 + x_2 \leq 20 &\quad \ldots \text{it corresponds to plain } p_2 \\
x_1 + x_2 \leq 12 &\quad \ldots \text{it corresponds to plain } p_3 \\
x_1, x_2 \geq 0 &
\end{aligned}
$$

Draw the set of feasible solutions and the optimal solution of the LPP graphically.
Solution:
Because all constraints are in the inequality form, we can draw them as half-planes p_1, p_2 a p_3. We obtain the set of feasible solutions as their intersection and also the intersection of half-planes expressing nonnegativity conditions (figure 2.1).

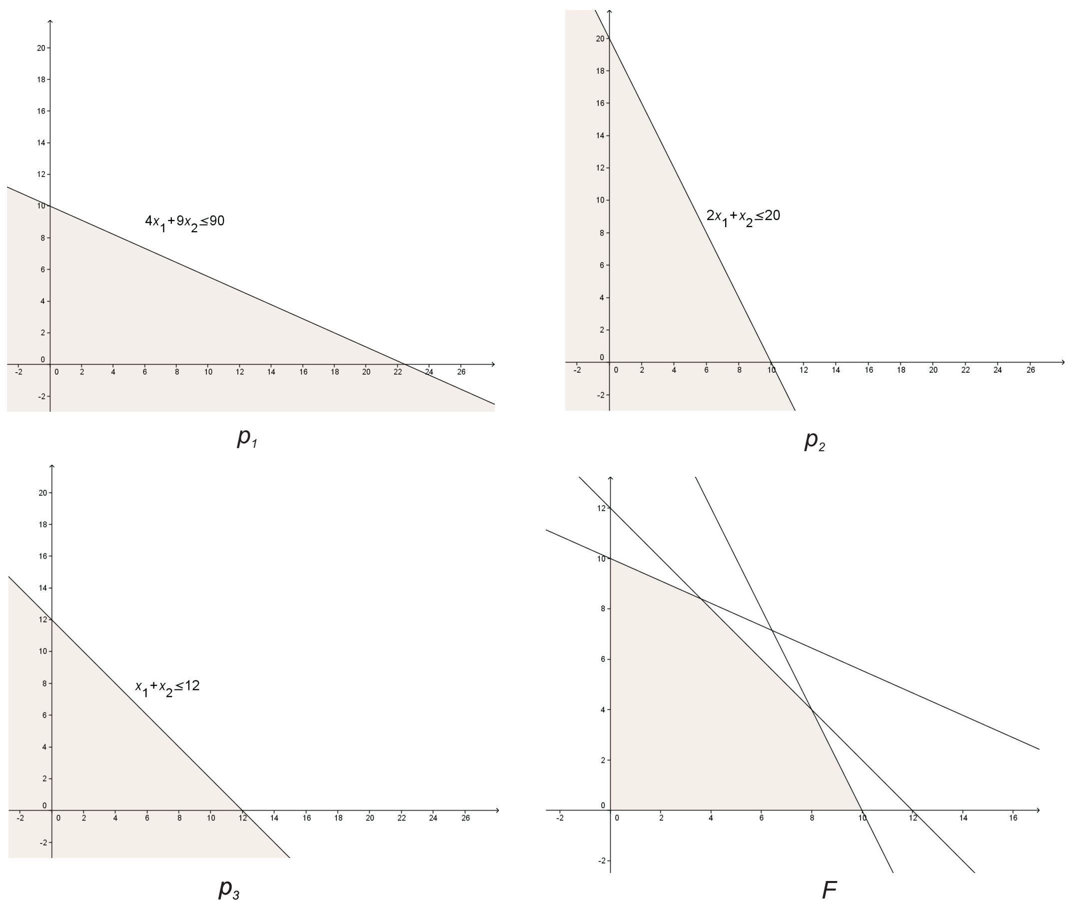

Figure 2.1: Constraints (p_1, p_2, p_3) and the set of feasible solutions F in $\mathbb{R}^2$.

We draw the contour line of the objective function – by the following way: Let $f(\boldsymbol{x}) = 0$, it means in our example, to draw the line p: $66x_1 + 48x_2 = 0$ and to denote in which direction the value of the objective function increases (figure 2.2).
We are looking for the furthest point of the set of feasible solutions F in the denoted direction (figure 2.2). It's the intersection of lines represented by the equations:

$$2x_1 + x_2 = 20; \quad x_1 + x_2 = 12$$

We gain the optimal solution coordinates by resolution of this equation system and we gain the optimal value of the objective function as the value of the objective function for

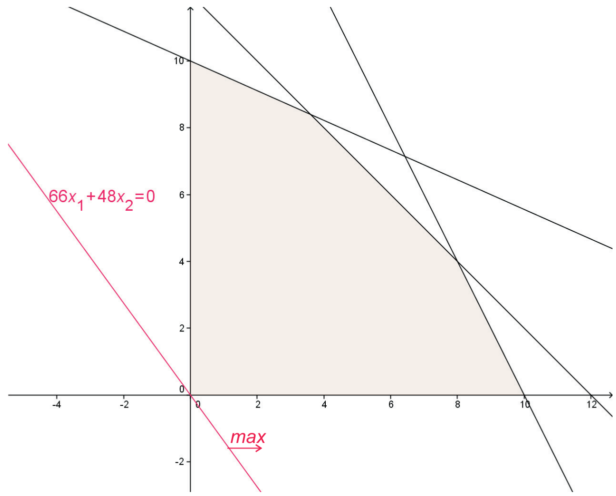

Figure 2.2: The contour line of the objective function.

the optimal solution:

$$\begin{aligned} \boldsymbol{x}^{opt} &= (8,4)^\top \\ f^{opt}(\boldsymbol{x}^{opt}) &= 720. \end{aligned}$$

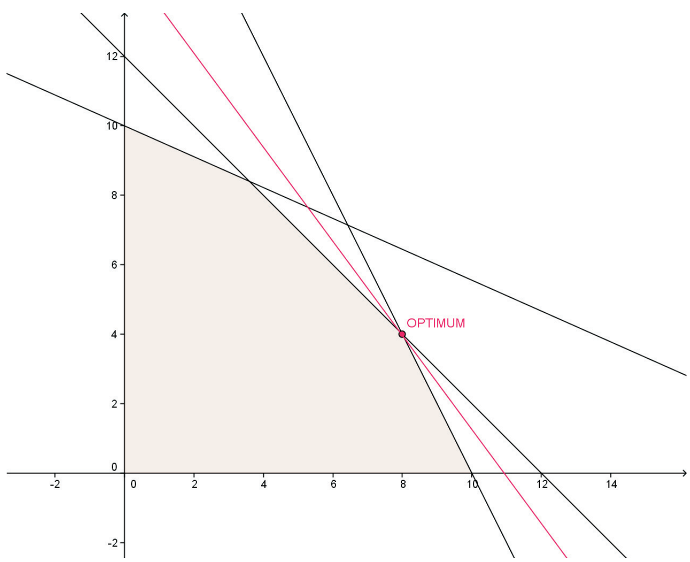

Figure 2.3: Optimum – the point at which the objective function has the maximum.

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Example 2.2. Let constraints of LPP be given by the following:

$$\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + 2x_2 &\geq 6 \\ 4x_1 - x_2 &\leq 15 \\ x_1 &\geq 0 \end{aligned}$$

Draw the set of feasible solutions and the optimal solution of the LPP graphically and find the optimal solution if the objective function is:

a) $f_1(\boldsymbol{x}) = x_1 + 2x_2 \to \max,$

b) $f_2(\boldsymbol{x}) = -x_1 + 3x_2 \to \min.$

Solution:
By the similar way as in the previous example 2.1, we draw the set of feasible solutions F as the intersection of half-planes. We can see on the figure 2.4, the set of feasible solutions is unbounded.

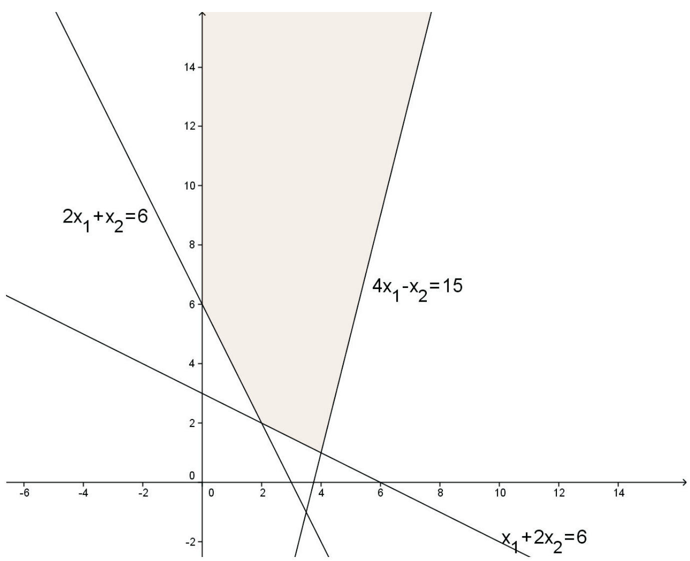

Figure 2.4: The set of feasible solutions.

Counter lines of objective functions $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{x})$ are drawn on figure 2.5. We can not find the optimum of $f_1(\boldsymbol{x})$ because the set is unbounded in the direction of maximization of the objective function, thus, the LPP is feasible, but unbounded.
Though, the objective function $f_2(\boldsymbol{x})$ has the optimal solution on the same feasible set. We compute the optimal solution and objective function value in this solution analogously as in the example 2.1:

$$\begin{aligned} \boldsymbol{x}^{opt} &= (4, 1)^\top \\ f_2^{opt}(\boldsymbol{x}^{opt}) &= -1. \end{aligned}$$

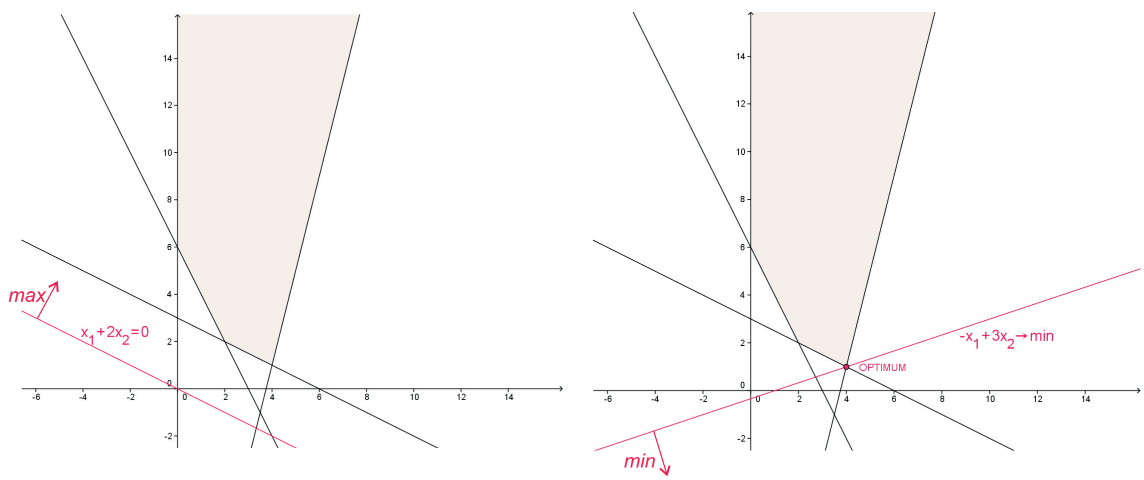

Figure 2.5: Counter lines of objective functions and the optimal solution.

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Example 2.3. Let the linear programming problem be given as follows. Draw the feasible set and the optimal solution of the LPP graphically.

$$\begin{aligned} f(\boldsymbol{x}) = -6x_1 + x_2 &\to \min \\ 6x_1 - x_2 &\leq 24 \\ 7x_1 + 2x_2 &\geq 14 \\ 4x_1 + 9x_2 &\leq 45 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:
On the figure 2.6, there are drawn the feasible set and the counter line of the objective function as line $-6x_1 + x_2 = 0$. We are approaching to the optimal solution by moving to the right. Since the boundary of the feasible set is given by condition $6x_1 - x_2 \leq 24$, this boundary is parallel to the contour line. The optimal solutions are all the points of the line segment $\overline{BC}$ and the number of optimal solutions is infinite. The value of the objective function for any of them is the same.

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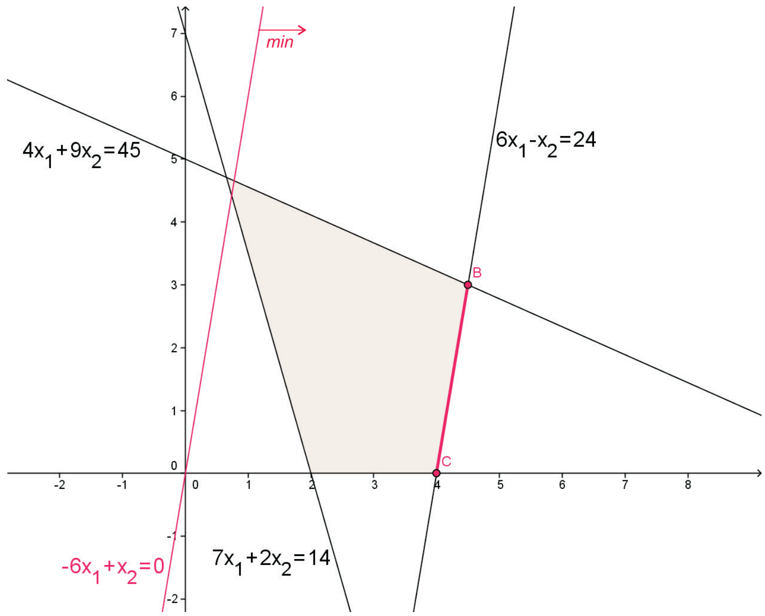

Figure 2.6: The feasible set, the counter line of the objective function and optimal solutions.

Example 2.4. A linear programming problem is given bellow. Draw the feasible set and the optimal solution of the LPP graphically.

$$\begin{aligned} f(\boldsymbol{x}) = x_1 + x_2 &\to \min \\ x_1 + x_2 &\leq 1 \\ 2x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:
Nonnegativity conditions should be valid for both variables. This means that the feasible set contain only the points from the first quadrant. Because the two half-planes for the constraints have no intersection in the first quadrant (see fig. 2.7), the feasible set is empty and LPP don't have an optimal solution.

$\surd$

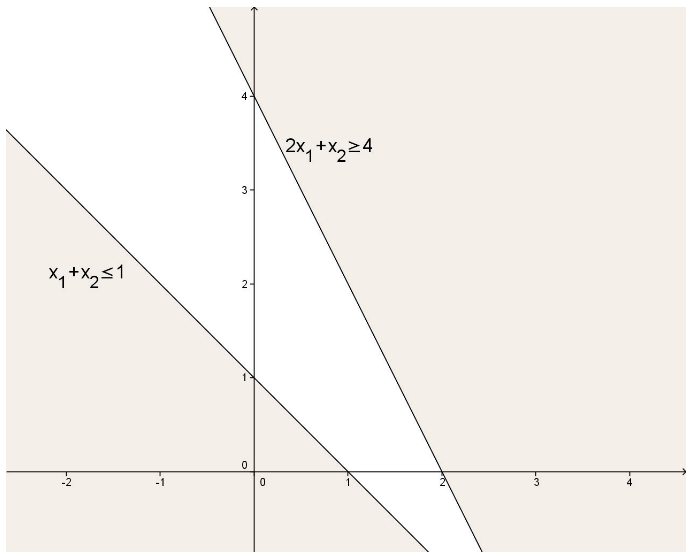

Figure 2.7: The feasible set.

Observation:
We might have noticed in the previous examples that the feasible set may be empty, non-empty bounded and non-empty unbounded. The number of optimal solutions could be zero, one, and infinity. The next table clearly shows, which options are possible ($\surd$) or aren't possible (–) for the pair "feasible set – number of optimal solutions".

Table 2.1: Information about the feasible set and its cardinality.

number of optimal solutions	feasible set		
	empty	non-empty bounded	non-empty unbounded
zero	$\surd$	–	$\surd$
one	–	$\surd$	$\surd$
infinity	–	$\surd$	$\surd$

2.4 The Introduction to Convex Analysis

Definition 2.4. A non-empty set $M \in \mathbb{R}^n$ is called a *convex set*, if:

$$(\forall \boldsymbol{x}, \boldsymbol{y} \in M)(\forall \lambda \in \langle 0; 1 \rangle) : (\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in M).$$

Let a set M be a subset of $\mathbb{R}^2$. If for every pair of points within the set M, every point on the straight line segment that joins the pair of points is also within the set M, then the set M is convex.

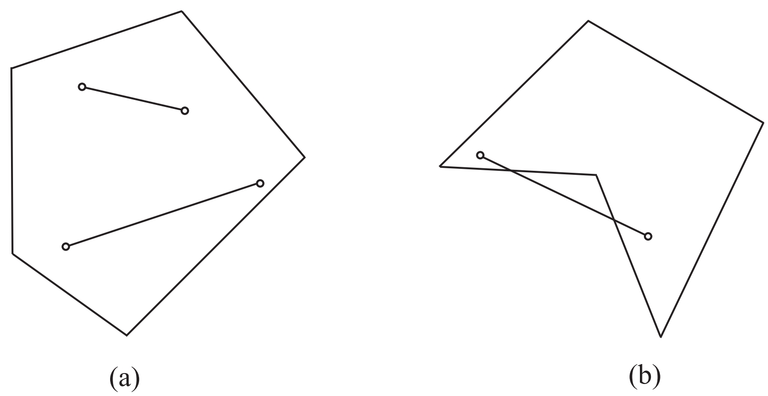

Figure 2.8: Example of a convex set (a) and a nonconvex set (b) in $\mathbb{R}^2$.

Theorem 2.1. The intersection of convex sets is a convex set.

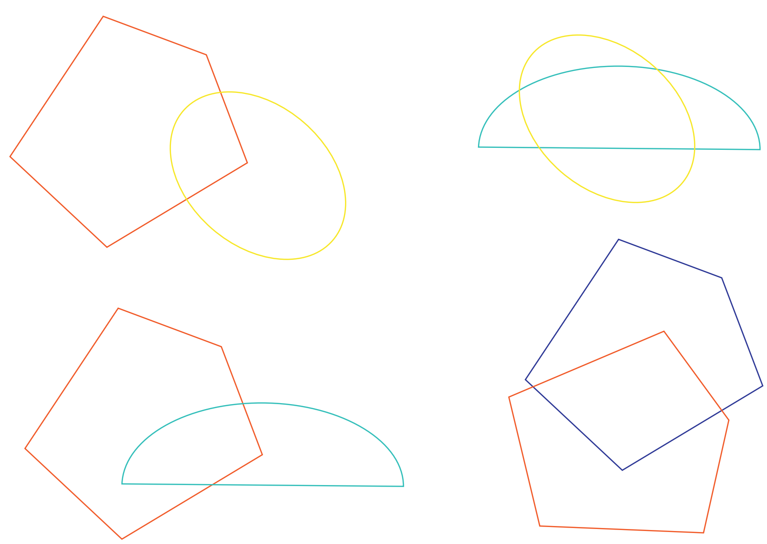

Figure 2.9: Intersections of pairs of convex sets in $\mathbb{R}^2$.

Definition 2.5. For any collection of points $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_k} \in \mathbb{R}^n$ and for any nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1,$$

the point $\boldsymbol{x} = \lambda_1 \boldsymbol{x_1} + \lambda_2 \boldsymbol{x_2} + \cdots + \lambda_k \boldsymbol{x_k} \in \mathbb{R}^n$ is called a convex combination of points $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_k}$.

Theorem 2.2. Consider a set $M \in \mathbb{R}^n$, $M \neq \emptyset$. A set M is a convex set iff any convex combination of any points of M belongs to M, too.

Definition 2.6. Let set $M \subseteq \mathbb{R}^n$. A set $conv(M)$ with the characteristic properties
(1) $M \subseteq conv(M)$,
(2) $conv(M)$ is convex set,
(3) If there exists covnex set M_1, such that $M \subseteq M_1$, then $conv(M) \subseteq M_1$,
is called a convex hull of the set M.

Consequently, the $conv(M)$ is the smallest convex set such that $M \subseteq conv(M)$.

Definition 2.7. Let M be a convex set. A point $\boldsymbol{x} \in M$ such that:
if $\boldsymbol{x} = \lambda \boldsymbol{y} + (1 - \lambda)\boldsymbol{z}$ for any $\boldsymbol{y}, \boldsymbol{z} \in M$ and $\lambda \in (0; 1)$, then $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$.
is called corner point of M.

Remark 2.2. There are used the denotation "boundary point, extreme point" for a corner point in literature.

Definition 2.8. A corner point of a set M is a point that can not be expressed as a non-trivial convex combination of points from M. A set of corner points of M is called $ex(M)$.

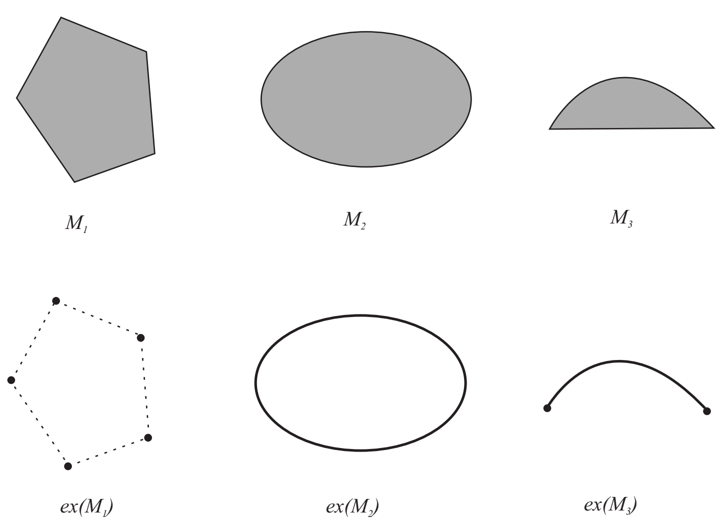

Figure 2.10: Convex sets and their sets of corner points.

Theorem 2.3. Every bounded, closed, convex and non-empty set contains at least one corner point.

Theorem 2.4. Let $M \subseteq \mathbb{R}^n$, $M \neq \emptyset$ be bounded, closed and a convex set. Then every $\boldsymbol{x} \in M$ can be expressed as a convex combination of corner points of the set M.

Definition 2.9. A closed convex set is called polyhedral set if it has a finite number of corner points.

Only the set M_1 is polyhedral in the picture 2.10.

Theorem 2.5. The feasible set F of any LPP is a convex set.

Theorem 2.6. The feasible set F of any LPP is polyhedral.

Theorem 2.7. The set of optimal solutions of any LPP is convex.

Theorem 2.8. Let us have a bounded and nonempty set F of feasible solutions of LPP. Then:

(1) there exists $min\{\boldsymbol{c}^\top . \boldsymbol{x} \ : \ \boldsymbol{x} \in F\} = f^*$

(2) there exists a corner point $\boldsymbol{x_0}$ of set F such that $\boldsymbol{c}^\top . \boldsymbol{x_0} = f^*$.

Theorem 2.9 (The main theorem of LPP)**.** There occurs exactly one of the following options for each minimization linear programming problem:

- LPP is infeasible, i.e., $F = \emptyset$.
- LPP is feasible but unbounded, i.e., $F \neq \emptyset$ and objective function $f(\boldsymbol{x}) = \boldsymbol{c}^\top \cdot \boldsymbol{x}$ is lower unbounded on F.
- LPP has an optimal solution in at least one of the corner points of the feasible set.

2.5 The Standard Form of Linear Programming Problem

Definition 2.10. We say that a LPP is in canonical form if it is in the form:

$$
\begin{aligned}
f(\boldsymbol{x}) = \sum_{j=1}^{n} (c_j \cdot x_j) \to \min \\
\sum_{j=1}^{n} (a_{ij} \cdot x_j) \le b_i, \quad \text{for } i = 1, \dots, m \\
x_j \ge 0, \quad \text{for } j = 1, 2, \dots, n.
\end{aligned}
\tag{2.4}
$$

Definition 2.11. We say that a LPP with n variables and m constraints is in standard form if it is in the form:

$$\begin{aligned} f(\boldsymbol{x}) = \sum_{j=1}^{n}(c_j \cdot x_j) &\to \min \\ \sum_{j=1}^{n}(a_{ij} \cdot x_j) = b_i, &\quad \text{for } i = 1, \dots, m \\ x_j \geq 0, &\quad \text{for } j = 1, 2, \dots, n. \end{aligned} \tag{2.5}$$

Remark 2.3. A matrix representation of a standard form of a LPP is:

$$\begin{aligned} & f(\boldsymbol{x}) = \boldsymbol{c}^\top \cdot \boldsymbol{x} \to \min \\ & \boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b} \\ & \quad x_j \geq 0, \quad \text{for } j = 1, 2, \dots, n. \end{aligned} \tag{2.6}$$

Theorem 2.10. A general, a canonical and a standard form of linear programming problems are equivalent to each other.

Thus, each LPP in general form can be transformed into a canonical or a standard form and vice versa, while the optimal solution does not change.

2.5.1 Conversions of LPP Forms

We use some fundamental transformations to convert linear programming problems from any form into other form:

1. The changing of maximization LPP to minimization (or vice versa) is realized by multiplying of the objective function by -1:

$$\begin{aligned} f(\boldsymbol{x}) &\to \max \quad / \cdot (-1) \\ -f(\boldsymbol{x}) &\to \min \end{aligned} \tag{2.7}$$

2. The conversion of constraint in inequality form "$\leq$" to constraint in inequality form "$\geq$" or vice versa is realized by multiplying the constraint by -1:

$$\begin{aligned} \sum_{j=1}^{n}(a_{ij} \cdot x_j) &\geq b_i \quad / \cdot (-1) \\ \sum_{j=1}^{n}(-a_{ij} \cdot x_j) &\leq -b_i \end{aligned} \tag{2.8}$$

3. The conversion of a constraint in an inequality form to a constraint in an equality form is realized by adding a nonnegative slack variables:

$$\begin{aligned} \sum_{j=1}^{n}(a_{ij} \cdot x_j) \geq b_i \quad &\rightarrow \quad \sum_{j=1}^{n}(a_{ij} \cdot x_j) - s_i = b_i; \quad s_i \geq 0 \\ \sum_{j=1}^{n}(a_{ij} \cdot x_j) \leq b_i \quad &\rightarrow \quad \sum_{j=1}^{n}(a_{ij} \cdot x_j) + s_i = b_i; \quad s_i \geq 0 \end{aligned} \tag{2.9}$$

4. The conversion of a constraint in an inequality form to a constraint in an equality form is realized by substitution the equation with two inequalities according to the principle of dichotomy:

$$\sum_{j=1}^{n}(a_{ij} \cdot x_j) = b_i \quad \rightarrow \quad \begin{array}{l} \sum_{j=1}^{n}(a_{ij} \cdot x_j) \leq b_i \\ \sum_{j=1}^{n}(a_{ij} \cdot x_j) \geq b_i \end{array} \tag{2.10}$$

5. The substitution of an infinite variable by a difference of two nonnegative variables:

$$x_j \text{ is unbounded} \quad \rightarrow \quad \begin{array}{l} x_j = x_j^{+} - x_j^{-} \\ x_j^{+} \geq 0; \; x_j^{-} \geq 0 \end{array} \tag{2.11}$$

   and:
   if $x_j = 0$, then $x_j^{+} = 0 \wedge x_j^{-} = 0$,
   if $x_j > 0$, then $x_j^{+} = x_j \wedge x_j^{-} = 0$,
   if $x_j < 0$, then $x_j^{+} = 0 \wedge x_j^{-} = -x_j$.

See example 2.11.

2.6 The Basis Feasible Solution of Linear Programming Problems

Let $\boldsymbol{A} \in \mathbb{R}_{m,n}$ be a matrix such that $h(\boldsymbol{A}) = m$ and $m \leq n$.[1] Considering properties of matrices, rows of $\boldsymbol{A}$ are linearly independent iff there exist m linearly independent columns in $\boldsymbol{A}$.

Definition 2.12. The set, which is created by m linearly independent columns of $\boldsymbol{A}$ is called a base of the matrix $\boldsymbol{A}$ and it is denoted $\mathcal{B}$. The matrix, which is created by columns of the base $\mathcal{B}$, is denoted $\boldsymbol{B}$.
Denotation: The base $\mathcal{B}$ is created by m linearly independent columns of the matrix $\boldsymbol{A}$, it will be denotated $\mathcal{B} = \{A_{B(1)}, A_{B(2)}, \ldots, A_{B(m)}\}$. So, $A_{B(i)}$ denotes a column of $\boldsymbol{A}$, which is i-th element of the base $\mathcal{B}$. $B(i)$ denotes the index of the column that is i-th element of

[1] $h(\boldsymbol{A})$ denotes the rank of the matrix $\boldsymbol{A}$

the base $\mathcal{B}$.
Clearly, the square matrix $\boldsymbol{B}$ is regular, thus there exists its inverse matrix $\boldsymbol{B}^{-1}$. See example 2.12.

Let $\boldsymbol{A} \in \mathbb{R}_{m,n}$ be a matrix with m linearly independent rows and n columns. A base of this matrix has to be $m \times m$. Hence, the maximum number of bases of $\boldsymbol{A}$ is $\binom{n}{m}$. See example 2.13.

Remark 2.4. Let $\mathcal{B} = \{A_{B(1)}, A_{B(2)}, \ldots, A_{B(m)}\}$ be a base of the matrix $\boldsymbol{A}$, then every column $A_j;\ j = 1, 2, \ldots, n$ of the matrix $\boldsymbol{A}$ can be expressed as a linear combination of basis columns:

$$A_j = \sum_{i=1}^{m} (x_{ij} \cdot A_{B(i)}),$$

and values x_{ij} are called coordinates of the column A_j in the base $\mathcal{B}$.

See example 2.14.

Definition 2.13. The solution $\boldsymbol{x}_{\mathcal{B}} = (x_1, x_2, \ldots, x_n)^\top$ of system $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ such that:

$$x_j = \begin{cases} 0; & if\ \boldsymbol{A}_j \notin \mathcal{B}, \\ \text{particular element (coordinate) of the solution of } \boldsymbol{B} \cdot \boldsymbol{x}_{\mathcal{B}} = \boldsymbol{b}; & if\ \boldsymbol{A}_j \in \mathcal{B}. \end{cases}$$

is called the basis solution (BS) of the given system for base $\mathcal{B}$.

It implies:

$$\sum_{i=1}^{m} (x_{B(i)} \cdot A_{B(i)}) = \boldsymbol{b}.$$

See example 2.15.

Definition 2.14. A basis solution $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^\top$ is called a basis feasible solution (BFS), if $x_j \geq 0$, $\forall j \in \{1, 2, \ldots, n\}$.

Definition 2.15. Let $\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b}$ be a system such that $\boldsymbol{A} \in \mathbb{R}_{m,n}$ is a matrix with m linearly independent columns. A basis solution with more than $n - m$ zero elements is called a degenerated solution.

See example 2.16.

Theorem 2.11. If two different bases correspond to the same basis solution $\boldsymbol{x}$, then this solution $\boldsymbol{x}$ is degenerated.

We can see it in the previous example. The solution $\boldsymbol{x}_{\mathcal{B}_1} = \boldsymbol{x}_{\mathcal{B}_2} = (2, 0, 0, 0)^\top$ is degenerated and it corresponds to two bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Similarly, the solution $\boldsymbol{x}_{\mathcal{B}_3} = \boldsymbol{x}_{\mathcal{B}_5} = (0, 0, 1, 0)^\top$ is also degenerated and it corresponds to two bases $\mathcal{B}_3$ and $\mathcal{B}_5$.

Theorem 2.12. The LPP in standard form (2.5) with matrix $\boldsymbol{A} \in \mathbb{R}_{m,n}$ has a basis feasible solution iff $F \neq \emptyset$ and $h(\boldsymbol{A}) = m$.

Theorem 2.13. If columns $A_1, A_2, \ldots, A_k$ of $\boldsymbol{A}$ are linearly independent and the solution $\boldsymbol{x} = (x_1, x_2, \ldots, x_k, 0, \ldots, 0, 0)^T \in F$, then $\boldsymbol{x} \in ex(F)$.

Theorem 2.14. If $\boldsymbol{x} \in ex(F)$, then the set $\{A_j, x_j \geq 0\}$ of matrix $\boldsymbol{A}$ columns is linearly independent set.

2.7 Solved Examples

Example 2.5. Iron foundry produces three different alloys (Z_1, Z_2, Z_3) for the aerospace industry, which arise by mixing four different metals (K_1, K_2, K_3, K_4) in precise proportions. We need $0,6$ kg of metal K_1 and $0,4$ kg of metal K_2 to produce one kilogram of alloy Z_1. One kilogram of alloy Z_2 consists of $0,5$ kg of metal K_2 and $0,5$ kg of metal K_4. One kilogram of alloy Z_3 consists of $0,3$ kg of metal K_3 and $0,7$ kg of metal K_4. Foundry has 5 kg of metal K_1, 6 kg of metal K_2, 7 kg of metal K_3 and 3 kg of metal K_4. Profit from the sale of one kilogram of alloy Z_1, Z_2 a Z_3 is 50 €, 40 € a 60 €. What production plan should be used to maximize its profits?
Solution:
We write the available data for the production of different types of alloys to a summary table :

Table 2.2: Summary Table – the weights, capacities and profits.

alloy\metal	K_1 (kg)	K_2 (kg)	K_3 (kg)	K_4 (kg)	profit (€)
Z_1 (1 kg)	0,6	0,4	0	0	50
Z_2 (1 kg)	0	0,5	0	0,5	40
Z_3 (1 kg)	0	0	0,3	0,7	60
capacity (kg)	5	6	7	3	

We write a mathematical model of the task using mathematical tools. Quantities of alloys which have to be produced by foundry are unknown, therefore it is decision variables of the objective function. Denote them x_1, x_2 and x_3. The objective function of this LPP is:

$$50x_1 + 40x_2 + 60x_3 \to \max .$$

Each alloy has a fixed ratio of metals. We also know that the amount of metals that is available is not unlimited, which means that we must not exceed the specified capacity. So we can write constrains in the inequalities form for each metal. For example K_2 is used to produce alloys Z_1 a Z_2. We need $0,4$ kg of it to produce one kilogram of Z_1 and $0,5$ kg of K_2 to produce one kilogram of Z_2. We have 6 kg of K_2. The constrain for K_2 is:

$$0,4x_1 + 0,5x_2 \leq 6$$

The nonnegativity constrains will be also included for each variable, because to consider the production of negative amount of alloys does not make sense.

$$x_1, x_2, x_3 \geq 0$$

The whole mathematical model of the LPP is as follows:

$$
\begin{aligned}
50x_1 + 40x_2 + 60x_3 &\rightarrow \max \\
0,6x_1 &\leq 5 \\
0,4x_1 + 0,5x_2 &\leq 6 \\
0,3x_3 &\leq 7 \\
0,5x_2 + 0,7x_3 &\leq 3 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

$\surd$

Example 2.6. A shipyard produces three types of ships: $L100$, $L80$ and $L40$. The ship $L100$ will earn 12 millions € for shipyard, the construction of this ship takes 6 months and it is able to transport 100 containers. The ship $L80$ will earn 10 millions € for the shipyard, the construction of this ship takes 4 months and it is able to transport 80 containers. The last type of the ship - $L40$ will earn 8 millions € for the shipyard, the construction of this ship takes 3 months and it is able to transport 40 containers. According to market research, the shipyard knows the fact it is possible to sell ships which are able to transport at most 320 containers, furthermore ships $L80$ are enough atypical, and therefore the shipyard has not sold more than 4, yet. Suggest a production plan for the next 20 months according to all the requirements and to get a maximum profit.
Solution:
The input data can be clearly written into the following table:

Table 2.3: Summary Table – capacities and profits

ships	time of construction [month]	transport capacity [pc]	profit [mil.€]
$L100$	6	100	12
$L80(\leq 4)$	4	80	10
$L40$	3	40	8
capacity	20	320	

Decision variables in this LPP will be the numbers of ships produced by each type $L100$, $L80$ and $L40$, we denote them as x_1, x_2 a x_3. The constrains are three, the first will be related to the time of construction, the second is related to the amount of containers and the third constrain will express the fact, that a limit for ships $L80$ is no more than 4 pieces. All three variables must be non-negative, moreover, the numbers of produced ships must be integers. Therefore, in this case, the integer condition will be added for all the decision variables:

$$x_1, x_2, x_3 \in \mathbb{Z}$$

The mathematical model of the LPP is as follows:

$$\begin{aligned} 12x_1 + 10x_2 + 8x_3 &\to \max \\ 6x_1 + 4x_2 + 3x_3 &\leq 20 \\ 100x_1 + 80x_2 + 40x_3 &\leq 320 \\ x_2 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \\ x_1, x_2, x_3 &\in \mathbb{Z} \end{aligned}$$

Such a problem is called the integer linear programming problem and denoted as ILPP. $\surd$

Example 2.7. A farmer keeps cattle on the farm. He has to buy the necessary amount of three offered semiproducts P_1, P_2, P_3 for its fattening. He finally mixed them and prepare a final dose of compound. This should include at least 5 kg of proteins, 7 kg of carbohydrates and $3,5$ kg of fat. There are 380 g of proteins, 240 g of carbohydrates and 200 g of fat in one kilogram of P_1. One kilogram of P_2 contains 180 g of proteins, 320 g of carbohydrates and 150 g of fat and one kilogram of P_3 contains 110 g of proteins, 220 g of carbohydrates, and 400 g of fat. Prices of semiproducts P_1, P_2 and P_3 per kilo are 4,30 €, 3,20 € and 3,70 €, respectively. The target is what quantities of each semiproducts is necessary to mix in order to reach a mixture with the required parameters while costs is minimal.
Solution:
We write data about the composition and prices of semiproducts into the table and we pay attention to the consistency of physical quantities:

Table 2.4: Summary Table – mixing problem

nutrient\semiproduct	P_1 (kg)	P_2 (kg)	P_3 (kg)	required amount (g)
proteins (g)	380	180	110	5000
carbohydrates (g)	240	320	220	7000
fat (g)	200	150	400	3500
price (€)	4,30	3,20	3,70	

By a similar way as in the previous examples we write the mathematical model of the problem. The objective function is a function of prices of semiproducts and each constraint (inequality) sets down the amount of proteins, carbohydrates and fat as they are entered in the table.

$$\begin{aligned}
4{,}3x_1 + 3{,}2x_2 + 3{,}7x_3 &\to \min \\
380x_1 + 180x_2 + 110x_3 &\geq 5000 \\
240x_1 + 320x_2 + 220x_3 &\geq 7000 \\
200x_1 + 150x_2 + 400x_3 &\geq 3500 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

$\surd$

Example 2.8. We have 18 bar pieces each with the length of 9 meters. We need to cut at least 8 bar pieces with the length of 5 meters, at least 14 bar pieces with the length of 4 meters and 20 bar pieces with the length of 3 meters. Suggest an optimal solution by minimizing the waste.
Solution:
Nine-meter bar can be cut to the required lengths in five ways:

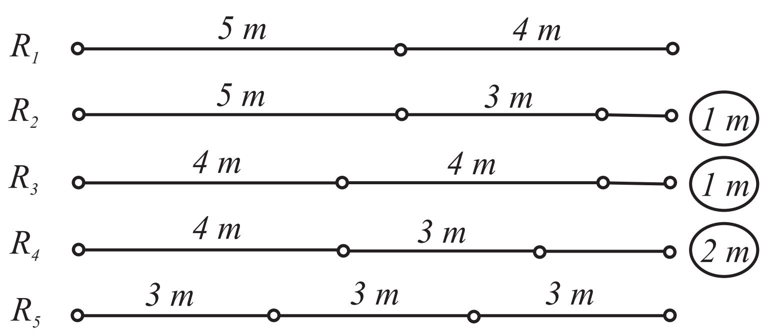

Figure 2.11: Possible Cutting Plans

We obtained a waste 1 meter by cuttings R_2 and R_3, the waste 2 meter by cuttings R_4 and no waste arises in cutting plan R_5. Thus, the number of variables is five and we will minimize the waste function. Constrains will be determined as the number of units required for any required length. The last condition considers the number of available bars. Of course, all variables must satisfy nonnegativity and integer conditions.

$$\begin{aligned}
x_2 + x_3 + 2x_4 &\to \min \\
x_1 + x_2 &\geq 8 \\
x_1 + 2x_3 + x_4 &\geq 14 \\
x_2 + x_4 + 3x_5 &\geq 20 \\
x_1 + x_2 + x_3 + x_4 + x_5 &\leq 18 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0 \\
x_1, x_2, x_3, x_4, x_5 &\in \mathbb{Z}
\end{aligned}$$

$\surd$

Example 2.9. A chain of hypermarkets has its central stores in KE, BA, and LM. These central stores dispose of the amounts 40, 40 and 20 units of the same item. The individual hypermarkets need these amounts of this item: TN – 25, ZA – 10, RV – 20, BB – 30, PP – 15. Transport costs of 1 unit of this item from central stores into individual hypermarkets are listed in the following table. Design a supply with this item in order to minimize the transport costs.

Table 2.5: List of distances between cities.

provider\customer	TN	ZA	RV	BB	PP
KE	55	60	30	50	40
BA	35	30	100	45	60
LM	40	30	95	35	30

Solution:
In this example we have five customers, so $n = 5$ and three providers, so $m = 3$. We will minimize the objective function (cost function), where the variable x_{ij} specifies the amount of units of the commodity to be transported from the i-th central store to the j-th hypermarket. Therefore, there are $m \cdot n = 3 \cdot 5 = 15$ variables. Objective function can be written as follows:

$$\begin{aligned} &55x_{11} + 60x_{12} + 30x_{13} + 50x_{14} + 40x_{15} + 35x_{21} + 30x_{22} + 100x_{23} + \\ &+45x_{24} + 60x_{25} + 40x_{31} + 30x_{32} + 95x_{33} + 35x_{34} + 30x_{35} \to \min \end{aligned}$$

We have the following constraints from the customer's requirements:

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} &= 40 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} &= 40 \\ x_{31} + x_{32} + x_{33} + x_{34} + x_{35} &= 20 \end{aligned}$$

We have the other following constraints from the provider's capacities:

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 25 \\ x_{12} + x_{22} + x_{32} &= 10 \\ x_{13} + x_{23} + x_{33} &= 20 \\ x_{14} + x_{24} + x_{34} &= 30 \\ x_{15} + x_{25} + x_{35} &= 15 \end{aligned}$$

It is evident that we cannot transport negative quantities of commodity, so the nonnegativity conditions must also be satisfied.

$$x_{ij} \geq 0 \qquad \text{for } i = 1, 2, 3; \quad j = 1, 2, 3, 4, 5$$

Because the equality

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{5} b_j = 100$$

is valid, the transportation problem is balanced. $\surd$

Example 2.10. A taxi service has 3 taxis (T_1, T_2, T_3) located at different places basis and they are available for assignment to 3 clients (C_1, C_2, C_3). Any taxi can be assigned to any client. The required time to move every taxi for eny client is given by the table below (in minutes). The taxi service wants to minimize the total time needed to transfer all three taxis to clients.

Table 2.6: The time data for the assignment problem.

taxi\client	C_1	C_2	C_3
T_1	13	15	20
T_2	14	10	17
T_3	12	15	12

Solution:
The task has nine variables, because $n = 3$. In a similar way as in the example 2.9, we write the objective function and constraints for taxis and clients. Only nonnegativity conditions are changed to conditions $x_{ij} \in \{0, 1\}$.

$$13x_{11} + 15x_{12} + 20x_{13} + 14x_{21} + 10x_{22} + 17x_{23} + 12x_{31} + 15x_{32} + 12x_{33} \to \min$$

$$\begin{array}{ccccccccccc}
x_{11}+ & x_{12}+ & x_{13} & & & & & & & = 1 \\
 & & & x_{21}+ & x_{22}+ & x_{23} & & & & = 1 \\
 & & & & & & x_{31}+ & x_{32}+ & x_{33} & = 1 \\
x_{11}+ & & & x_{21}+ & & & x_{31} & & & = 1 \\
 & x_{12}+ & & & x_{22}+ & & & x_{32} & & = 1 \\
 & & x_{13}+ & & & x_{23}+ & & & x_{33} & = 1
\end{array}$$

$$x_{ij} \in \{0,1\} \quad \text{for } i,j = 1,2,\dots,n$$

$\surd$

Example 2.11. Transform the given LPP to a canonical and standard form:

$$\begin{aligned} x_1 - x_2 &\to \max \\ 3x_1 - 5x_2 &\le 8 \\ -2x_1 + x_2 &\ge 4 \\ x_1 + x_2 &= 6 \\ x_1 &\ge 0 \end{aligned}$$

Solution:
The objective function have to be minimization, so we use fundamental transformation (2.7):

$$-x_1 + x_2 \to \min$$

The variable x_2 is unbounded. We substitute it by using transformation (2.11):

$$x_2 = x_2^+ - x_2^-; \quad x_2^+ \ge 0;\ x_2^- \ge 0$$

and we obtain:

$$\begin{aligned} -x_1 + x_2^+ - x_2^- &\to \min \\ 3x_1 - 5x_2^+ + 5x_2^- &\le 8 \\ -2x_1 + x_2^+ - x_2^- &\ge 4 \\ x_1 + x_2^+ - x_2^- &= 6 \\ x_1, x_2^+, x_2^- &\ge 0 \end{aligned} \tag{2.12}$$

We need all constraints in inequality form "$\ge$" for canonical form. We multiply the first constraint by -1 and we use the transformation (2.10) for the third constraint and then we multiply obtained constraint by -1. The canonical form is:

$$\begin{aligned} -x_1 + x_2^+ - x_2^- &\to \min \\ -3x_1 + 5x_2^+ - 5x_2^- &\ge -8 \\ -2x_1 + x_2^+ - x_2^- &\ge 4 \\ x_1 + x_2^+ - x_2^- &\ge 6 \\ -x_1 - x_2^+ + x_2^- &\ge -6 \\ x_1, x_2^+, x_2^- &\ge 0 \end{aligned}$$

We use (2.12) to make the standard form. We add slack variables into 1. and 2. constraint with using (2.9). The standard form of LPP is:

$$\begin{aligned} -x_1 + x_2^+ - x_2^- &\to \min \\ 3x_1 - 5x_2^+ + 5x_2^- + s_1 &= 8 \\ -2x_1 + x_2^+ - x_2^- - s_1 &= 4 \\ x_1 + x_2^+ - x_2^- &= 6 \\ x_1, x_2^+, x_2^-, s_1, s_2 &\geq 0 \end{aligned}$$

$\surd$

Example 2.12. Let's have given the following matrix $\boldsymbol{A}$. We want to choose its base.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 4 & 3 & 2 \\ 3 & 1 & 5 & 4 & 2 \\ 1 & 0 & 2 & 1 & 1 \end{pmatrix}$$

Solution:
We choose columns A_1, A_2 a A_5 from matrix $\boldsymbol{A}$. They are linearly independent, so they create a base say $\mathcal{B}_1$. Similarly, columns A_2, A_3 a A_4 also create a base say $\mathcal{B}_2$. Let matrices $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ be created by columns of these bases. The matrix $\boldsymbol{C_1}$, which consists of A_1, A_2 and A_4, is not a matrix of a base, because $A_4 = A_1 + A_2$. Similarly the matrix $\boldsymbol{C_2}$, which consists of A_1, A_3 and A_5, is not a matrix of the base, because $A_3 = A_5 + A_1$.

$$\boldsymbol{B}_1 = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{B}_2 = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 5 & 4 \\ 0 & 2 & 1 \end{pmatrix},$$

$$\boldsymbol{C}_1 = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{C}_2 = \begin{pmatrix} 2 & 4 & 2 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

$\surd$

Example 2.13. Find all bases of the matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 4 & -1 \\ 1 & 5 & 2 & 0 \end{pmatrix}$$

Solution:
Rows of the $\boldsymbol{A}$ are linearly independent, so, rank of this matrix is $h(\boldsymbol{A}) = 2$. The number of columns of $\boldsymbol{A}$ is 4. Thus, the maximum number of bases of $\boldsymbol{A}$ is $\binom{4}{2} = 6$. We create all 6 possible submatrices with size 2×2 from columns of $\boldsymbol{A}$.

$$\boldsymbol{M}_1 = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}, \quad \boldsymbol{M}_2 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \quad \boldsymbol{M}_3 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{M}_4 = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}, \quad \boldsymbol{M}_5 = \begin{pmatrix} 3 & -1 \\ 5 & 0 \end{pmatrix}, \quad \boldsymbol{M}_6 = \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}.$$

Columns of $\boldsymbol{M}_2$, which is created by columns A_1 and A_3, are linearly independent (one is a multiple of another), therefore this matrix cannot be a base matrix of $\boldsymbol{A}$. In all other cases, the columns are linearly independent and the matrices are base matrices of the matrix $\boldsymbol{A}$:

$$\mathcal{B}_1 = \{A_1, A_2\}, \quad \mathcal{B}_2 = \{A_1, A_4\}, \quad \mathcal{B}_3 = \{A_2, A_3\}, \quad \mathcal{B}_4 = \{A_2, A_4\}, \quad \mathcal{B}_5 = \{A_3, A_4\}.$$

$\surd$

Example 2.14. Calculate the coordinates of the column A_4 in the base $\mathcal{B}_3$ from the example 2.13.

Solution:

The base $\mathcal{B}_3$ is created by columns A_2, A_3. According to the remark 2.4, we write:

$$A_4 = x_{14} \cdot A_2 + x_{24} \cdot A_3.$$

We obtain the following system by substituting of particular elements of columns:

$$\begin{aligned} -1 &= 3x_{14} + 4x_{24} \\ 0 &= 5x_{14} + 2x_{24} \end{aligned}$$

We solve system and we obtain coordinates of the column A_4 in the base $\mathcal{B}_3$: $\boldsymbol{x}^{\mathcal{B}_3} = \left(\frac{1}{7}, -\frac{5}{14}\right)$

$\surd$

Example 2.15. Find all basis solutions of the system

$$\begin{pmatrix} 2 & 3 & 4 & -1 \\ 1 & 5 & 2 & 0 \end{pmatrix} \cdot \boldsymbol{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Solution:

We denote the matrix on the left side of the system as $\boldsymbol{A}$. It is the same matrix as in the example 2.13 and therefore there exist five different bases of $\boldsymbol{A}$

$$\mathcal{B}_1 = \{A_1, A_2\}, \quad \mathcal{B}_2 = \{A_1, A_4\}, \quad \mathcal{B}_3 = \{A_2, A_3\}, \quad \mathcal{B}_4 = \{A_2, A_4\}, \quad \mathcal{B}_5 = \{A_3, A_4\}.$$

We denote the matrix (column) on the right side of the system by $\boldsymbol{b}$. We calculate a solution of $\boldsymbol{B}_k \cdot \boldsymbol{x} = \boldsymbol{b}$ for every base $\mathcal{B}_k$, $k \in \{1, 2, 3, 4, 5\}$. According to the defiinition 2.13, we write all basis solutions $\boldsymbol{x}_{\mathcal{B}_k}$.

$$\mathcal{B}_1 = \{A_1, A_2\}; \quad \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \boldsymbol{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}; \quad \boldsymbol{x}_{\mathcal{B}_1} = (2, 0, 0, 0)^\top$$

$$\mathcal{B}_2 = \{A_1, A_4\}; \quad \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \cdot \boldsymbol{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}; \quad \boldsymbol{x}_{\mathcal{B}_2} = (2, 0, 0, 0)^\top$$

$$\mathcal{B}_3 = \{A_2, A_3\}; \quad \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} \cdot \boldsymbol{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}; \quad \boldsymbol{x}_{\mathcal{B}_3} = (0, 0, 1, 0)^\top$$

$$\mathcal{B}_4 = \{A_2, A_4\}; \quad \begin{pmatrix} 3 & -1 \\ 5 & 0 \end{pmatrix} \cdot \boldsymbol{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}; \quad \boldsymbol{x}_{\mathcal{B}_4} = \left(0, \frac{2}{5}, 0, -\frac{14}{5}\right)^\top$$

$$\mathcal{B}_5 = \{A_3, A_4\}; \quad \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} \cdot \boldsymbol{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}; \quad \boldsymbol{x}_{\mathcal{B}_5} = (0, 0, 1, 0)^\top$$

$\surd$

Example 2.16. Determine which of the solutions in the example 2.15 are feasible and which are degenerated.
Solution:
Basis feasible solutions: $\boldsymbol{x}_{\mathcal{B}_1}$, $\boldsymbol{x}_{\mathcal{B}_2}$, $\boldsymbol{x}_{\mathcal{B}_3}$ and $\boldsymbol{x}_{\mathcal{B}_5}$.
Degenerated solutions are the same $\boldsymbol{x}_{\mathcal{B}_1}$, $\boldsymbol{x}_{\mathcal{B}_2}$, $\boldsymbol{x}_{\mathcal{B}_3}$ and $\boldsymbol{x}_{\mathcal{B}_5}$. $\surd$

2.8 Exercises

Design a mathematical model for the verbally formulated linear programming problems 2.1– 2.9.

2.1. We have 30 bar pieces each with the length of 12 meters. We need to cut 15 bar pieces with the length of 5 meters, 40 bar pieces with the length of 4 meters and 35 bar pieces with the length of 3 meters. Suggest a optimal solution by minimizing the scrap.

2.2. There is a cutting machine available for a cutting line which is able to cut standardized bales with the width of 2 meters. From these standardized bales we have to cut the required amount of bales with following widths: 862 pc by 112 cm, 341 pc by 77 cm and 216 pc by 35 cm. Let us assume that we have sufficient number of the standardized bales and we cut only the required widths (of course scrap will be caused by this). Suggest the setting of the cutting tools in the cutting machine (and also their presetting) so that the scrap would be minimized.

2.3. You have 12 000 $ to invest, and three different funds from which you can choose. The municipal bond fund (MBF) has a 7% return, the local bank's CDs have an 8% return, and the high-risk account has an expected (hoped-for) 12% return. To minimize risk, you decide not to invest more than 2, 000 $ in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. Assuming the year-end yields are as expected, what are optimal investment amounts?

2.4. At a certain refinery, the refining process requires the production of at least three gallons of gasoline for each gallon of fuel oil. To meet the anticipated demands of winter, at least three million gallons of fuel oil a day will need to be produced. The demand for gasoline, on the other hand, is not more than 6.4 million gallons a day. If gasoline is selling for $ 4.50 per gallon and fuel oil sells for $ 5.90/gal, how much of each should be produced in order to maximize revenue?

2.5. A farmer has 10 acres to plant in wheat and rye. He has to plant at least 7 acres. However, he has only $ 1 200 to invest and each acre of wheat costs $ 200 to plant and each acre of rye costs $ 100 to plant. Moreover, the farmer has to get the planting done in 12 hours and it takes an hour to plant an acre of wheat and 2 hours to plant an acre of rye. If the profit is $ 500 per acre of wheat and $ 300 per acre of rye how many acres of each should be planted to maximize profits?

2.6. A gold processor has two sources of gold ore, source A and source B. In order to keep his plant running, at least three tons of ore must be processed each day. Ore from source A costs $ 1 000 per ton to process, and ore from source B costs $ 2 000 per ton to process. Costs must be at most $ 8 000 per day. Moreover, Federal Regulations require that the amount of ore from source B cannot exceed twice the amount of ore from source A. If ore from source A yields 2 oz. of gold per ton, and ore from source B yields 3 oz. of gold per ton, how many tons of ore from both sources must be processed each day to maximize the amount of gold extracted subject to the above constraints?

2.7. A chain of hypermarkets has its central stores in BA, LM and KE. These central stores dispose of the amounts 40, 20 and 40 units of the same item. The individual hypermarkets need these amounts of this item: TN – 25, ZA – 20, BB – 30, PP – 25. Transport costs of 1 unit of this item from central stores into hypermarkets are listed in the following table. Design a supply with this item in order to minimize the transport costs.

	TN	ZA	BB	PP
KE	55	60	50	40
BA	35	30	45	60
LM	40	30	35	30

2.8. A carpenter makes tables and chairs and he wants to have a maximal profit. Each table can be sold for a profit of £30 and each chair for a profit of £10. The carpenter can afford to spend up to 40 hours per week working and takes six hours to make a table and three hours to make a chair. Customer demand requires that he makes at least three times as many chairs as tables. Tables take up four times as much storage space as chairs and there is room for at most four tables each week. Formulate this problem as a linear programming problem.

2.9. A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators much be shipped each day. If each scientific calculator sold results in a \$ 2 loss, but each graphing calculator produces a \$ 5 profit, how many of each type should be made daily to maximize net profits?

2.10. Convert the following linear programming problems into canonical and standard form.

a)

$$\begin{aligned} 2x_1 + x_2 &\to \max \\ 4x_1 - x_2 &\leq 4 \\ x_1 + 2x_2 &\geq 5 \\ x_1 &\leq 6 \\ x_{1,2} &\geq 0 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 - 3x_2 &\to \min \\ x_1 + x_2 &= 15 \\ x_1 - x_2 &\geq 7 \\ 3x_1 + x_2 &\leq 3 \\ x_{1,2} &\geq 0 \end{aligned}$$

c)

$$\begin{aligned} x_1 - 4x_2 + x_3 &\to \max \\ x_1 + x_2 + x_3 &= 10 \\ 2x_1 + x_2 + 3x_3 &\leq 120 \\ x_{1,2,3} &\geq 0 \end{aligned}$$

d)

$$\begin{aligned} 4x_1 - x_2 + 3x_3 &\to \min \\ 2x_1 - 8x_2 + 2x_3 &= 6 \\ 3x_1 - 4x_2 + 3x_3 &\geq 4 \\ 7x_1 + 5x_2 + x_3 &\geq -4 \\ x_{1,2} &\geq 0 \end{aligned}$$

e)

$$\begin{aligned} -x_1 - 2x_2 - 3x_3 &\to \max \\ x_1 - x_2 + 4x_3 &\leq 6 \\ x_1 + 2x_2 - 3x_3 &\geq 7 \\ x_1 - 2x_3 &= 3 \\ x_{2,3} &\geq 0 \end{aligned}$$

f)

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &\to \min \\ 2x_1 + 3x_2 - x_4 &= 8 \\ 4x_1 - 7x_3 + 2x_4 &= 12 \end{aligned}$$

2.9 Solutions

2.1

$$\begin{aligned}
2x_1 + x_3 + x_5 + 2x_6 &\to \min \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &\le 30 \\
2x_1 + x_2 + x_3 &\ge 15 \\
x_2 + 3x_4 + 2x_5 + x_6 &\ge 35 \\
x_2 + 2x_3 + x_5 + 2x_6 + 4x_7 &\ge 40 \\
x_{1,\dots,7} &\ge 0
\end{aligned}$$

2.2

$$\begin{aligned}
11x_1 + 18x_2 + 11x_3 + 18x_4 + 25x_5 &\to \min \\
x_1 + x_2 &\ge 862 \\
x_1 + 2x_3 + x_4 &\ge 341 \\
2x_2 + x_3 + 3x_4 + 5x_5 &\ge 216 \\
x_{1,\dots,5} &\ge 0
\end{aligned}$$

2.3

$$\begin{aligned}
0,07x_1 + 0,08x_2 + 0,12x_3 &\to \max \\
x_1 + x_2 + x_3 &= 12000 \\
x_3 &\le 2000 \\
x_1 - 3x_2 &\ge 0 \\
x_{1,2,3} &\ge 0
\end{aligned}$$

2.4

$$\begin{aligned}
4,5x_1 + 5,9x_2 &\to \max \\
x_1 - 3x_2 &\ge 0 \\
x_1 &\ge 3 \\
0 \le x_2 &\le 6,4
\end{aligned}$$

2.5

$$\begin{aligned}
500x_1 + 300x_2 &\to \max \\
x_1 + x_2 &\le 10 \\
x_1 + x_2 &\ge 7 \\
200x_1 + 100x_2 &\le 1200 \\
x_1 + 2x_2 &\le 12 \\
x_{1,2} &\ge 0
\end{aligned}$$

2.6

$$\begin{aligned} 60x_1 + 90x_2 &\to \max \\ x_1 + x_2 &\geq 3 \\ 2000x_1 + 1000x_2 &\leq 8000 \\ 2x_1 - x_2 &\geq 0 \\ x_{1,2} &\geq 0 \end{aligned}$$

2.7

$$\begin{aligned} 55x_{11} + 60x_{12} + 50x_{13} + 40x_{14} + 35x_{21} + 30x_{22} + 45x_{23} + 60x_{24} + \\ +40x_{31} + 30x_{32} + 35x_{33} + 30x_{34} &\to \min \\ x_{11} + x_{12} + x_{13} + x_{14} &= 40 \\ x_{21} + x_{22} + x_{23} + x_{24} &= 40 \\ x_{31} + x_{32} + x_{33} + x_{34} &= 20 \\ x_{11} + x_{21} + x_{31} &= 25 \\ x_{12} + x_{22} + x_{32} &= 20 \\ x_{13} + x_{23} + x_{33} &= 30 \\ x_{14} + x_{24} + x_{34} &= 25 \\ x_{11,12,\ldots,34} &\geq 0 \end{aligned}$$

2.8

$$\begin{aligned} 30x_1 + 10x_2 &\to \max \\ 6x_1 + 3x_2 &\leq 40 \\ 3x_1 - x_2 &\leq 0 \\ 4x_1 + x_2 &\leq 16 \\ x_{1,2} &\geq 0 \end{aligned}$$

2.9

$$\begin{aligned} -2x_1 + 5x_2 &\to \max \\ 100 \leq x_1 &\leq 200 \\ 80 \leq x_2 &\leq 170 \\ x_1 + x_2 &\geq 200 \\ x_{1,2} &\geq 0 \end{aligned}$$

2.10

canonical form | standard form

a)

$$\begin{aligned} -2x_1 - x_2 &\to \min \\ -4x_1 + x_2 &\geq -4 \\ x_1 + 2x_2 &\geq 5 \\ -x_1 &\geq -6 \\ x_{1,2} &\geq 0 \end{aligned} \qquad \begin{aligned} -2x_1 - x_2 &\to \min \\ 4x_1 - x_2 + s_1 &= 4 \\ x_1 + 2x_2 - s_2 &= 5 \\ x_1 + s_3 &= 6 \\ x_{1,2} &\geq 0 \\ s_{1,2,3} &\geq 0 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 - 3x_2 &\to \min \\ -x_1 - x_2 &\geq -15 \\ x_1 + x_2 &\geq 15 \\ x_1 - x_2 &\geq 7 \\ -3x_1 - x_2 &\geq -3 \\ x_{1,2} &\geq 0 \end{aligned} \qquad \begin{aligned} 2x_1 - 3x_2 &\to \min \\ x_1 + x_2 &= 15 \\ x_1 - x_2 - s_1 &= 7 \\ 3x_1 + x_2 + s_2 &= 3 \\ x_{1,2} &\geq 0 \\ s_{1,2} &\geq 0 \end{aligned}$$

c)

$$\begin{aligned} -x_1 + 4x_2 - x_3 &\to \min \\ -x_1 - x_2 - x_3 &\geq -10 \\ x_1 + x_2 + x_3 &\geq 10 \\ -2x_1 - x_2 - 3x_3 &\geq -120 \\ x_{1,2,3} &\geq 0 \end{aligned} \qquad \begin{aligned} -x_1 + 4x_2 - x_3 &\to \min \\ x_1 + x_2 + x_3 &= 10 \\ 2x_1 + x_2 + 3x_3 + s_1 &= 120 \\ x_{1,2,3} &\geq 0 \\ s_1 &\geq 0 \end{aligned}$$

d)

$$\begin{aligned} 4x_1 - x_2 + 3x_3^+ - 3x_3^- &\to \min \\ -2x_1 + 8x_2 - 2x_3^+ + 2x_3^- &\geq -6 \\ 2x_1 - 8x_2 + 2x_3^+ - 2x_3^- &\geq 6 \\ 3x_1 - 4x_2 + 3x_3^+ - 3x_3^- &\geq 4 \\ 7x_1 + 5x_2 + x_3^+ - x_3^- &\geq -4 \\ x_{1,2}, x_3^+, x_3^- &\geq 0 \end{aligned} \qquad \begin{aligned} 4x_1 - x_2 + 3x_3^+ - 3x_3^- &\to \min \\ 2x_1 - 8x_2 + 2x_3^+ - 2x_3^- &= 6 \\ 3x_1 - 4x_2 + 3x_3^+ - 3x_3^- - s_1 &= 4 \\ 7x_1 + 5x_2 + x_3^+ - x_3^- - s_2 &= -4 \\ x_{1,2}, x_3^+, x_3^- &\geq 0 \\ s_{1,2} &\geq 0 \end{aligned}$$

e)

$$\begin{aligned} x_1^+ - x_1^- + 2x_2 + 3x_3 &\to \min \\ -x_1^+ + x_1^- + x_2 - 4x_3 &\geq -6 \\ x_1^+ - x_1^- + 2x_2 - 3x_3 &\geq 7 \\ -x_1^+ + x_1^- + 2x_3 &\geq 3 \\ x_1^+ - x_1^- - 2x_3 &\geq 3 \\ x_1^+, x_1^-, x_{2,3} &\geq 0 \end{aligned}$$

$$\begin{aligned} x_1^+ - x_1^- + 2x_2 + 3x_3 &\to \min \\ x_1^+ - x_1^- - x_2 + 4x_3 + s_1 &= 6 \\ x_1^+ - x_1^- + 2x_2 - 3x_3 - s_2 &= 7 \\ x_1^+ - x_1^- - 2x_3 &= 3 \\ x_1^+, x_1^-, x_{2,3} &\geq 0 \\ s_{1,2} &\geq 0 \end{aligned}$$

f)

$$\begin{aligned} x_1^+ - x_1^- - x_2^+ + x_2^- + x_3^+ - x_3^- - x_4^+ + x_4^- &\to \min \\ -2x_1^+ + 2x_1^- - 3x_2^+ + 3x_2^- + x_4^+ - x_4^- &\geq -8 \\ 2x_1^+ - 2x_1^- + 3x_2^+ - 3x_2^- - x_4^+ + x_4^- &\geq 8 \\ -4x_1^+ + 4x_1^- + 7x_3^+ - 7x_3^- - 2x_4^+ + 2x_4^- &\geq -12 \\ 4x_1^+ - 4x_1^- - 7x_3^+ + 7x_3^- + 2x_4^+ - 2x_4^- &\geq 12 \\ x_1^+, x_1^-, x_2^+, x_2^-, x_3^+, x_3^-, x_4^+, x_4^- &\geq 0 \end{aligned}$$

$$\begin{aligned} x_1^+ - x_1^- - x_2^+ + x_2^- + x_3^+ - x_3^- - x_4^+ + x_4^- &\to \min \\ 2x_1^+ - 2x_1^- + 3x_2^+ - 3x_2^- - x_4^+ + x_4^- &= 8 \\ 4x_1^+ - 4x_1^- - 7x_3^+ + 7x_3^- + 2x_4^+ - 2x_4^- &= 12 \\ x_1^+, x_1^-, x_2^+, x_2^-, x_3^+, x_3^-, x_4^+, x_4^- &\geq 0 \end{aligned}$$

Chapter 3

Linear Programming Duality

3.1 The Dual to Linear Programming Problem

Consider the following LP with n variables and m constraints:

$$\begin{aligned}
\boldsymbol{c}^\top \cdot \boldsymbol{x} &\to \min \\
\boldsymbol{a_i} \cdot \boldsymbol{x} &= b_i, \quad \text{for } i = 1, \dots, k-1 \\
\boldsymbol{a_i} \cdot \boldsymbol{x} &\geq b_i, \quad \text{for } i = k, \dots, m \\
x_j &\geq 0 \quad \text{for } j \in N_1 \\
x_j &\text{ is unbounded} \quad \text{for } j \in N_2.
\end{aligned} \tag{3.1}$$

We could transform LPP to this form very easily by using basic transformations (2.7) – (2.11).

Definition 3.1. Consider the following LP in the form (3.1). The linear programming problem given by the following way is called a dual LPP (D) of original LPP. (3.1).

$$\begin{aligned}
\boldsymbol{y}^\top \cdot \boldsymbol{b} &\to \max \\
y_i &\text{ is unbounded} \quad \text{for } i = 1, \dots, k-1 \\
y_i &\geq 0 \quad \text{for } i = k, \dots, m \\
\boldsymbol{y}^\top \cdot A_j &\leq c_j \quad \text{for } j \in N_1 \\
\boldsymbol{y}^\top \cdot A_j &= c_j \quad \text{for } j \in N_2.
\end{aligned} \tag{3.2}$$

The original LPP is called the primal problem (P).

If the primal LPP is in the canonical form, then pair primal - dual is given as:

$$\begin{aligned}
\boldsymbol{c}^\top \cdot \boldsymbol{x} &\to \min & \qquad \boldsymbol{y}^\top \cdot \boldsymbol{b} &\to \max \\
\boldsymbol{A} \cdot \boldsymbol{x} &\geq \boldsymbol{b} & \qquad \boldsymbol{y}^\top \cdot \boldsymbol{A} &\leq \boldsymbol{c} \\
\boldsymbol{x} &\geq 0 & \qquad \boldsymbol{y} &\geq 0
\end{aligned}$$

If the primal LPP is in the standard form, then pair primal - dual is given by:

$$
\begin{aligned}
\boldsymbol{c}^\top \cdot \boldsymbol{x} &\to \min & \qquad \boldsymbol{y}^\top \cdot \boldsymbol{b} &\to \max \\
\boldsymbol{A} \cdot \boldsymbol{x} &= \boldsymbol{b} & \qquad \boldsymbol{y}^\top \cdot \boldsymbol{A} &\leq \boldsymbol{c} \\
\boldsymbol{x} &\geq 0 & \qquad y_i &\text{ is unbounded} \quad \text{for } i = 1, \ldots, m.
\end{aligned}
$$

Theorem 3.1. Dual of a dual is primal.
Some basic rules for constructing of a dual.

1. if P is maximum (minimum) problem, then D is minimum (maximum) problem,
2. one variable in D belongs to one constraint in P,
3. one constraint in D belongs to one variable in P,
4. coefficients of the objective function of P give corresponding right side in D,
5. elements of the right side in P give coefficients of the objective function in D,
6. the constraint matrix of D is transpose of the constraint matrix of P.

We clearly summarize these rules and signs of equality and inequality in primal-dual pair in the Table 3.1.

See examples 3.1 and 3.2.

3.2 Primal-Dual Solutions

Theorem 3.2 (The weak duality theorem)**.** For any feasible solution $\boldsymbol{x}$ in P (3.1) and feasible solution $\boldsymbol{y}$ in D (3.2) we have:

$$\boldsymbol{c}^\top \cdot \boldsymbol{x} \geq \boldsymbol{y}^\top \cdot \boldsymbol{b}.$$

Corollary 3.1. If $\boldsymbol{x}$ is feasible solution for P (3.1), $\boldsymbol{y}$ is feasible solution for D (3.2) such that:

$$\boldsymbol{c}^\top \cdot \boldsymbol{x} = \boldsymbol{y}^\top \cdot \boldsymbol{b},$$

then both $\boldsymbol{x}, \boldsymbol{y}$ are optimal for their respective LPP.

Corollary 3.2. If the feasible set of dual F_D (3.2) isn't empty and its objective function is uper unbounded on F_D, then the primal LPP (3.1) is infeasible.

Corollary 3.3. If the feasible set of primal F_P (3.1) isn't empty and its objective function is lower unbounded on F_P, then the dual LPP (3.2) is infeasible.

Table 3.1: Relations between primal (P) and dual (D) task of LPP.

Primal LPP (P)	$\Longleftrightarrow$	Dual LPP (D)
$\boldsymbol{c}^\top \cdot \boldsymbol{x} \to \min$	$\Longleftrightarrow$	$\boldsymbol{y}^\top \cdot \boldsymbol{b} \to \max$
$\boldsymbol{c}^\top \cdot \boldsymbol{x} \to \max$	$\Longleftrightarrow$	$\boldsymbol{y}^\top \cdot \boldsymbol{b} \to \min$
$\boldsymbol{a_i} \cdot \boldsymbol{x} \geq b_i$ (min)	$\Longleftrightarrow$	$y_i \geq 0$ (max)
$\boldsymbol{a_i} \cdot \boldsymbol{x} \geq b_i$ (max)	$\Longleftrightarrow$	$y_i \leq 0$ (min)
$\boldsymbol{a_i} \cdot \boldsymbol{x} \leq b_i$ (min)	$\Longleftrightarrow$	$y_i \leq 0$ (max)
$\boldsymbol{a_i} \cdot \boldsymbol{x} \leq b_i$ (max)	$\Longleftrightarrow$	$y_i \geq 0$ (min)
$\boldsymbol{a_i} \cdot \boldsymbol{x} = b_i$	$\Longleftrightarrow$	$y_i \in (-\infty, \infty)$
$x_j \geq 0$ (max)	$\Longleftrightarrow$	$\boldsymbol{y}^\top \cdot A_j \geq c_j$ (min)
$x_j \leq 0$ (max)	$\Longleftrightarrow$	$\boldsymbol{y}^\top \cdot A_j \leq c_j$ (min)
$x_j \geq 0$ (min)	$\Longleftrightarrow$	$\boldsymbol{y}^\top \cdot A_j \leq c_j$ (max)
$x_j \leq 0$ (min)	$\Longleftrightarrow$	$\boldsymbol{y}^\top \cdot A_j \geq c_j$ (max)
$x_j \in (-\infty, \infty)$	$\Longleftrightarrow$	$\boldsymbol{y}^\top \cdot A_j = c_j$

Theorem 3.3 (The strong duality theorem)**.**

(1) If either P or D has an optimal solution, then so does the other, the optimal values of objective functions are equal, and there exists optimal solutions for both P and D.

(2) If either P or D is feasible but unbounded, then the other is unfeasible.

Overview of the different options for solving a pair P – D:

Table 3.2: Overview of the different options for solving a pair P – D.

primal	dual		
	has optimum	feasible unbounded	infeasible
has optimum	$\surd$	–	–
feasible unbounded	–	–	$\surd$
infeasible	–	$\surd$	$\surd$

Theorem 3.4 (The complementary slackness theorem)**.** Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be feasible solution for P and D respectively. Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions if, and only if:

$$y_i(\boldsymbol{a_i} \cdot \boldsymbol{x} - b_i) = 0 \quad \text{for } i = 1, \ldots, m,$$

and

$$(c_j - \boldsymbol{y}^\top \cdot A_j)x_j = 0 \quad \text{for } j = 1, \ldots, n.$$

See examples 3.3 and 3.4.

3.3 Solved Examples

Example 3.1. Find the dual to the following primal LPP:

$$\begin{array}{rrrrl}
x_1 & +x_2 & -3x_3 & +x_4 & \to \min \\
3x_1 & -2x_2 & -x_3 & & \leq 4 \\
 & x_2 & +x_3 & +4x_4 & \leq 2 \\
x_1 & & +3x_3 & & \geq 3 \\
 & & & x_{1-4} & \geq 0
\end{array}$$

Solution:
Clearly, the dual is a maximum problem, because of minimum primal. The primal contains 4 variables (x_1, x_2, x_3, x_4) and 3 constraints, therefore the dual contains 4 constraints and 3 variables (y_1, y_2, y_3). Using rules 1. - 6. we could write what we have determined so far:

$$\begin{array}{rrrr}
4y_1 & +2y_2 & +3y_3 & \to \max \\
3y_1 & & +y_3 & 1 \\
-2y_1 & +y_2 & & 1 \\
-y_1 & +y_2 & +3y_3 & -3 \\
 & 4y_2 & & 1
\end{array}$$

We determine signs of equality and inequality with respect to the above table. So we have a complete mathematical model of the desired dual LPP:

$$\begin{array}{rrrl}
4y_1 & +2y_2 & +3y_3 & \to \max \\
3y_1 & & +y_3 & \leq 1 \\
-2y_1 & +y_2 & & \leq 1 \\
-y_1 & +y_2 & +3y_3 & \leq -3 \\
 & 4y_2 & & \leq 1
\end{array}$$

$$y_1, y_2 \leq 0; \quad y_3 \geq 0.$$

$\surd$

Example 3.2. Find the dual to the following primal LPP:

$$\begin{array}{rrrl}
2x_1 & -x_2 & +4x_3 & \to \max \\
x_1 & +3x_2 & -2x_3 & \geq 0 \\
2x_1 & +2x_2 & +4x_3 & \leq 6 \\
x_1 & -x_2 & -x_3 & = -8 \\
 & & x_1 & \geq 0.
\end{array}$$

Solution:
Similarly as in the previous example 3.1, we write the coefficients of the objective function of P as right side coefficients of D, elements of right side of P as coefficients of the objective function of D a constraint matrix of D will be transpose of the constraint matrix of P:

$$\begin{array}{rrrc}
 & 6y_2 & -8y_3 & \to \min \\
y_1 & +2y_2 & +y_3 & 2 \\
3y_1 & +2y_2 & -y_3 & -1 \\
-2y_1 & +4y_2 & -y_3 & 4
\end{array}$$

According to known rules listed in Table, we determine signs of equality and inequality in constraints:

$$\begin{array}{rrrl}
 & 6y_2 & -8y_3 & \to \min \\
y_1 & +2y_2 & +y_3 & \geq 2 \\
3y_1 & +2y_2 & -y_3 & = -1 \\
-2y_1 & +4y_2 & -y_3 & = 4 \\
 & & y_1 & \leq 0 \\
 & & y_2 & \geq 0 \\
 & & y_3 & \in (-\infty, \infty).
\end{array}$$

$\surd$

Example 3.3. Find the optimal solution of the given LPP:

$$\begin{aligned}
30x_1 + 48x_2 + 12x_3 &\to \min \\
3x_1 + 4x_2 - 2x_3 &= 1 \\
5x_1 + 3x_2 + 3x_3 &\geq -2 \\
x_1, x_3 &\geq 0 \\
x_2 &\leq 0
\end{aligned}$$

Solution:
The dual problem of this problem has two variables and three constraints. We can solve LPP with two variables graphically. So, first we write the dual problem of the given LPP:

$$\begin{aligned} y_1 - 2y_2 &\to \max \\ 3y_1 + 5y_2 &\leq 30 \\ 4y_1 + 3y_2 &\geq 48 \\ -2y_1 + 3y_2 &\leq 12 \\ y_2 &\geq 0 \end{aligned}$$

We represent constraints as half-planes in $\mathbb{R}^2$, see figure 3.1. When we add the nonnega-

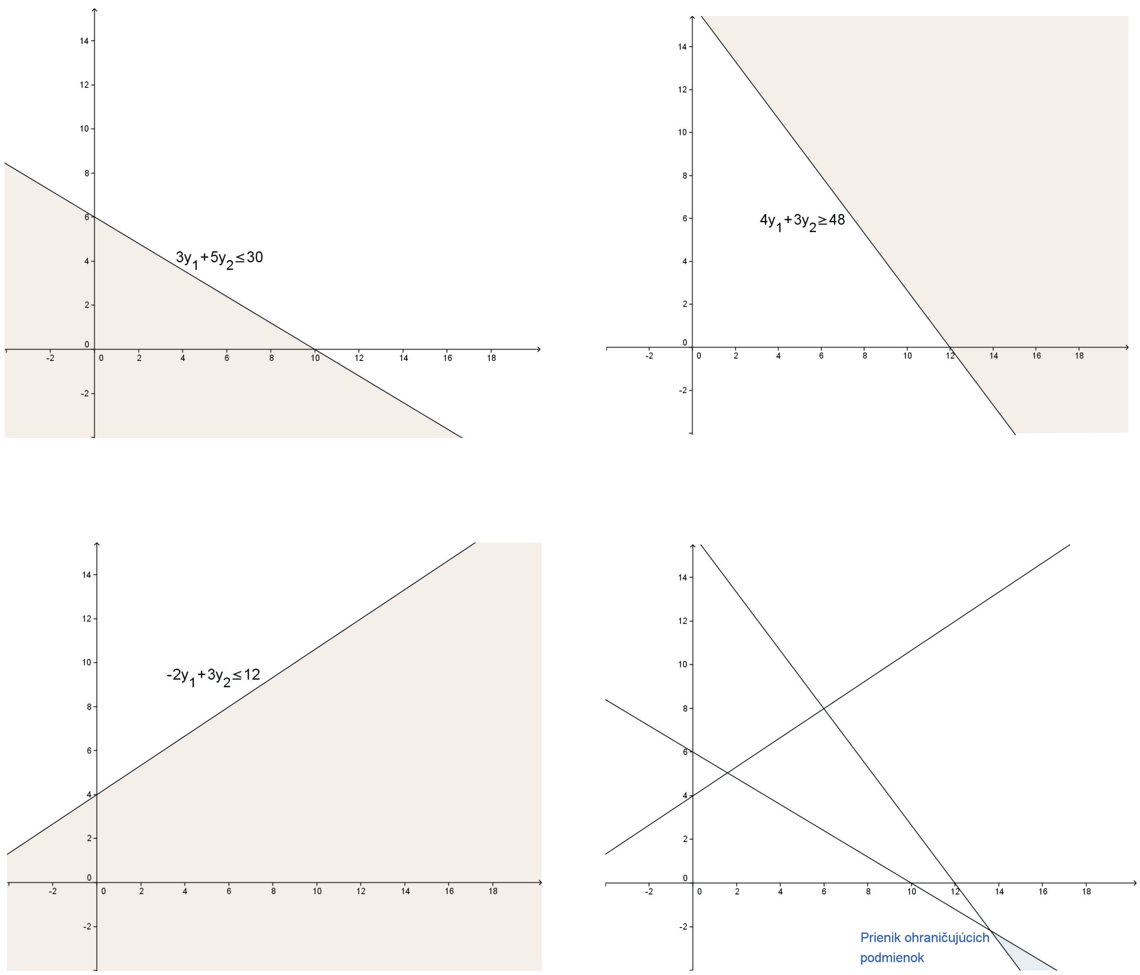

Figure 3.1: Constraints for dual LPP.

tivity condition for the variable y_2, we get the empty set of feasible solutions. Since the dual LPP is unfeasible, according to Theorem 3.3 we know that the primal LPP doesn't have the optimal solution. $\surd$

Example 3.4. Find the optimal solution of the given LPP. Use a similar procedure as in the example 3.3:

$$\begin{aligned} 3x_1 - x_2 + 2x_3 + x_4 &\to \min \\ x_1 + x_2 + x_3 + x_4 &\geq -1 \\ x_1 - x_2 + x_3 - x_4 &\geq 3 \\ x_{1-4} &\geq 0 \end{aligned}$$

Solution:
Similarly as in example 3.3, we have the primal with 4 variables and 3 constraints, therefore the dual contains 4 constraints and 3 variables and we know to solve it graphically. Mathematical model of dual is:

$$\begin{aligned} -y_1 + 3y_2 &\to \max \\ y_1 + y_2 &\leq 3 \\ y_1 - y_2 &\leq -1 \\ y_1 + y_2 &\leq 2 \\ y_1 - y_2 &\leq 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

We draw the feasible set and the counter line of the objective function of dual graphically. See figure 3.2

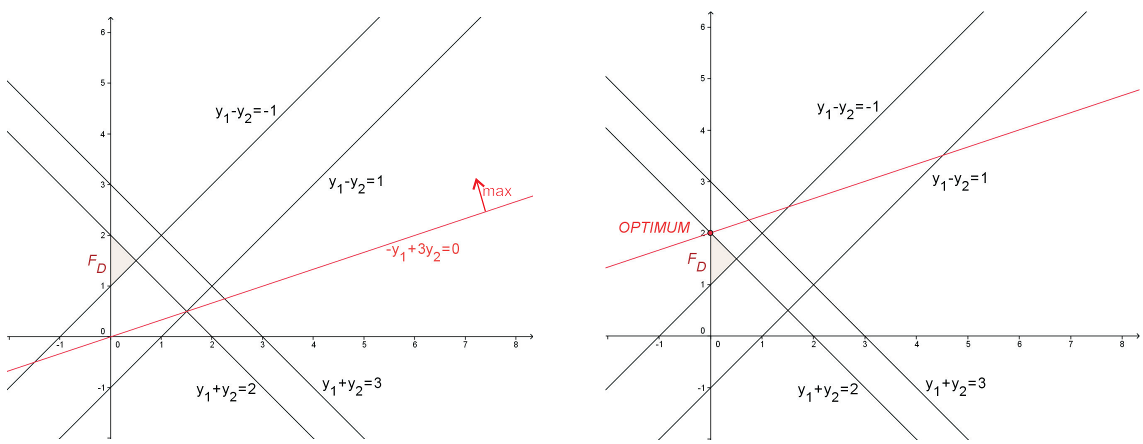

Figure 3.2: The graphic solution of the dual.

We obtain the optimal solution by moving of the counter line in the maximization direction: $\boldsymbol{y}^{opt} = (0,2)^\top$, $f_D^{opt}(\boldsymbol{y}) = 6$. By the strong duality theorem, the value of the primal objective function is $f_P^{opt}(\boldsymbol{x}) = f_D^{opt}(\boldsymbol{y}) = 6$.
We use the complementary slackness theorem to find the optimal solution of the primal. First, we substitute the $\boldsymbol{y}^{opt}$ into all constraints of dual and we find out which inequality

is acquired as a sharp inequality:

$$\begin{aligned} 0+2 &\le 3 & 2 &< 3 \\ 0-2 &\le -1 & -2 &< -1 \\ 0+2 &\le 2 & 2 &\le 2 \\ 0-2 &\le 1 & -2 &< 1 \end{aligned}$$

According to the complementary slackness theorem we know: $(c_j - \boldsymbol{y}^\top \cdot A_j)x_j = 0$. For the constrain to acquire sharp, it is $(c_j - \boldsymbol{y}^\top \cdot A_j) \neq 0$, hence $x_j = 0$ and $x_1 = x_2 = x_4 = 0$. Now we apply the second part of the complementary slackness theorem: $y_i(\boldsymbol{a_i} \cdot \boldsymbol{x} - b_i) = 0$. We know, that $y_2 \neq 0$, consequently, the second constraint of primal should be acquired as equality. Thus we substitute into this constraint $x_1 = x_2 = x_4 = 0$ and we compute x_3.

$$\begin{aligned} 0 - 0 + x_3 - 0 &= 3 \\ x_3 &= 3 \end{aligned}$$

The optimal solution of primal is $\boldsymbol{x}^{opt} = (0, 0, 3, 0)^\top$. $\surd$

3.4 Exercises

3.1. Find the dual to the following primal LPP::

$$\begin{aligned} 4x_1 - x_2 + 2x_3 &\to \max \\ -2x_1 + 3x_2 + x_3 &\geq 1 \\ 4x_1 + 2x_2 + 2x_3 &\leq 6 \\ -x_1 - x_2 + x_3 &= 8 \\ x_1 &\geq 0 \end{aligned}$$

3.2. Find the dual to the following primal LPP and convert it into standard form:

$$\begin{aligned} -3x_1 + 4x_2 + 2x_3 - x_4 &\to \min \\ 4x_1 - x_2 + 2x_3 - x_4 &\geq 1 \\ -2x_1 + x_2 + 2x_3 + 4x_4 &= 7 \\ -x_1 - x_2 + x_3 &\leq 8 \\ x_{1,4} &\geq 0 \end{aligned}$$

3.3. Let us have a primal LPP:

$$\begin{aligned} 12x_1 + 8x_2 &\to \min \\ 2x_1 + x_2 &\geq 4 \\ 2x_1 + 3x_2 &\geq 8 \\ x_1 + 6x_2 &\geq 6 \\ x_{1,2} &\geq 0 \end{aligned}$$

According to the complementary slackness theorem, find the optimal solution of the given primal, if we know the solution of its dual $\boldsymbol{y}^{opt} = (5, 1, 0)^\top$.

3.4. Let us have a primal LPP:

$$\begin{aligned} x_1 - x_2 &\to \max \\ x_1 + 3x_2 &\geq 9 \\ x_1 + 2x_2 &\leq 14 \\ -x_1 + 2x_2 &\leq 3 \\ x_1 &\leq 6 \end{aligned}$$

According to the complementary slackness theorem, find the optimal solution of the dual, if we know the solution of the given primal $\boldsymbol{x}^{opt} = (6; 1)^\top$.

3.5. Let us have a primal LPP:

$$\begin{aligned} 2x_1 - x_2 &\to \max \\ -3x_1 + x_2 &\le 9 \\ 5x_1 - x_2 &\le 6 \\ 3x_1 - x_2 &\ge 1 \\ 2x_1 + x_2 &\le 3 \\ x_{1,2} &\ge 0 \end{aligned}$$

According to the complementary slackness theorem, find the optimal solution of the given primal, if we know the solution of its dual $\boldsymbol{y}^{opt} = (0, 2/5, 0, 0)^\top$.

3.6. Let us have a primal LPP:

$$\begin{aligned} 11x_1 + 108x_2 - 45x_3 - 10x_4 &\to \max \\ x_1 + 26x_2 + 2x_3 - 2x_4 &\le 9 \\ 4x_1 + 15x_2 - 9x_3 + x_4 &\le 5 \\ x_{1,2,3,4} &\ge 0 \end{aligned}$$

Solve the dual graphically and use it to determine the solution of the given primal (with using the complementary slackness theorem).

3.7. Let us have a primal LPP:

$$\begin{aligned} x_1 + 6x_2 + 5x_3 &\to \min \\ -x_1 - 2x_2 + x_3 &\ge 1 \\ x_1 - 3x_2 + 10x_3 &\le -2 \\ x_{1,2,3} &\ge 0 \end{aligned}$$

Solve the dual graphically and use this solution to determine the solution of the given primal.

3.8. Let us have a primal LPP:

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &\to \max \\ x_1 + x_2 - 2x_3 &\le 5 \\ 2x_1 - 3x_2 + x_3 &\le 10 \\ x_{1,2,3} &\ge 0 \end{aligned}$$

Solve the dual graphically and use it to determine the solution of the given problem.

3.5 Solutions

3.1

$$\begin{aligned} y_1 + 6y_2 + 8y_3 &\to \min \\ -2y_1 + 4y_2 - y_3 &\geq 4 \\ 3y_1 + 2y_2 - y_3 &= -1 \\ y_1 + 2y_2 + y_3 &= 2 \\ y_1 \leq 0; y_2 &\geq 0 \end{aligned}$$

3.2

$$\begin{aligned} -y_1 - 7y_2^+ + 7y_2^- + 8y_3 &\to \min \\ 4y_1 - 2y_2^+ + 2y_2^- + y_3 + s_1 &= -3 \\ -y_1 + y_2^+ - y_2^- + y_3 &= 4 \\ 2y_1 + +2y_2^+ - 2y_2^- - y_3 &= 2 \\ -y_1 + 4y_2^+ - 4y_2^- + s_2 &= -1 \\ y_1, y_2^+, y_2^-, y_3, s_1, s_2 &\geq 0 \end{aligned}$$

3.3 $\boldsymbol{x}^{opt} = (1; 2)^\top,\ f(\boldsymbol{x})^{opt} = 28$

3.4 $\boldsymbol{y}^{opt} = (-1/3; 0; 0; 4/3)^\top,\ f(\boldsymbol{y})^{opt} = 5$

3.5 $\boldsymbol{x}^{opt} = (6/5; 0)^\top,\ f(\boldsymbol{x})^{opt} = 12/5$

3.6 $\boldsymbol{y}^{opt} = (9/8; 21/4)^\top$, $\boldsymbol{x}^{opt} = (0; 91/264; 5/264; 0)^\top,\ f(\boldsymbol{x})^{opt} = 291/8$

3.7 The dual is feasible unbounded, so the given primal is unfeasible.

3.8 $\boldsymbol{y}^{opt} = (2 - 2t; t)^\top$, for $t \in \langle 0, 1\rangle$ (line segment); $\boldsymbol{x}^{opt} = (5; 0; 0)^\top,\ f(\boldsymbol{x})^{opt} = 10$

Chapter 4

Simplex Method

4.1 Simplex Method – Algorithm

Simplex method is an algorithm how we can find an optimal solution of a linear programming problem if it exists. The simplex method is a deterministic algorithm to find out whether any base exists at all and find at least one basis feasible solution. The simplex method systematically scans the basis feasible solutions such that the algorithm:

(1) never returns to a basis feasible solution already visited,

(2) finds out that the linear programming problem is unbounded,

(3) finds an optimal solution of LPP.

Let LPP be in standard form and x_0 be the basis feasible solution corresponding to base $\mathcal{B} = \{A_{B(i)}; i = 1, 2, \ldots, m\}$. We determine the number:

$$\theta = \min_{i=1,\ldots,m} \left\{ \frac{x_{i0}}{x_{ij}}; \text{where} \quad x_{ij} > 0 \right\}. \tag{4.1}$$

Suppose that the minimum was achieved in row r such that $B(r) = k$. We obtain new base $\mathcal{B}^N = \mathcal{B} \cup \{j\} - \{k\}$, where

$$B^N(i) = \begin{cases} B(i) & for \quad i \neq r \\ j & for \quad i = r \end{cases} \tag{4.2}$$

and new basis feasible solution x_{i0}^N:

$$x_{i0}^N = \begin{cases} x_{i0} - \theta \cdot x_{ij} & for \quad i \neq r \\ \theta & for \quad i = r \end{cases}. \tag{4.3}$$

Transition between basis feasible solutions is called *pivoting*, the element x_{rj} is the *pivot*, the column A_j enters the base in position r and column $A_{B(r)} = A_k$ leaves base.

Simplex Table:

Denote:

$$z_0 = \sum_{i=1}^{m} x_{i0} \cdot c_{B(i)}, \tag{4.4}$$

$$z_j = \sum_{i=1}^{m} x_{ij} \cdot c_{B(i)}, \tag{4.5}$$

for each $j = 1, 2, \ldots, n$. Number z_0 is the value of the objective function in basis feasible solution x_0 and c_j^R is called a relative price of column A_j and it holds that $c_j^R = c_j - z_j$.

Table 4.1: Simplex Method – Simplex Table

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6
—	$-z_0$	c_1^R	c_2^R	c_3^R	c_4^R	c_5^R	c_6^R
A_1	x_{10}	1	0	0	0	a_{14}	a_{15}
A_2	x_{20}	0	1	0	0	a_{24}	a_{25}
A_3	x_{30}	0	0	1	0	a_{34}	a_{35}
A_4	x_{40}	0	0	0	1	a_{44}	a_{45}

Theorem 4.1 (Change in objective function value)**.** If a pivoting step is performed in basis feasible solution x_0 such that column A_j enters base, change in the objective function value is $\theta \cdot c_j^R = \theta \cdot (c_j - z_j)$.

Theorem 4.2. If there exists a column j with negative relative price $c_j^R = c_j - z_j < 0$, then on its entering the base the objective function value decreases by $\theta \cdot c_j^R = \theta \cdot (c_j - z_j)$.

Theorem 4.3 (Optimality criterion)**.** If vector $\boldsymbol{c}^R = \boldsymbol{c} - \boldsymbol{z}$ is nonnegative, then basis feasible solution x_0 is optimal.

Theorem 4.4 (Criterion of unboundedness)**.** If there exists column j with $c_j^R < 0$ such that for each $i:\ x_{ij} \leqq 0$, then the LPP is unbounded.

Remark 4.1. Let B is a square matrix corresponding to base $\mathcal{B}$. We can express the basis feasible solution x_0 for base $\mathcal{B}$ as:

$$\boldsymbol{x}_0 = B^{-1} \cdot \boldsymbol{b}, \tag{4.6}$$

and the coefficients of the j-th column for the given matrix A and base B as:

$$\boldsymbol{x}_j = B^{-1} \cdot A_j. \tag{4.7}$$

We can write:

$$z_0 = \boldsymbol{c}_B^\top \cdot \boldsymbol{x}_0 = \boldsymbol{c}_B^\top \cdot B^{-1} \cdot \boldsymbol{b} \tag{4.8}$$

$$z_j = \boldsymbol{c}_B^\top \cdot \boldsymbol{x}_j = \boldsymbol{c}_B^\top \cdot B^{-1} \cdot A_j, \quad \boldsymbol{z} = \boldsymbol{c}_B^\top \cdot B^{-1} \cdot A. \tag{4.9}$$

4.2 Two-Phase Algorithm of Simplex Method

As we can see in the previous section, the simplex method can be use for the linear programming problem in standard form, which the simplex table is primarily feasible (i. e. in the zero column are non-negative values) and the matrix limitation $\boldsymbol{A}$ contains m-dimensional unit sub-matrix, which forms a normal base.

If the matrix $\boldsymbol{A}$ does not contain identity sub-matrix, we use a two-phase algorithm of the simplex method, where the first phase is called the artificial LP (the auxiliary tasks). Let LPP be given in standard form. (2.5):

$$f(\boldsymbol{x}) = \sum_{j=1}^{n}(c_j \cdot x_j) \to \min$$

$$\sum_{j=1}^{n}(a_{ij} \cdot x_j) = b_i, \quad \text{for } i = 1, \dots, m$$

$$x_j \geq 0, \quad \text{for } j = 1, 2, \dots, n.$$

First phase: It consists of solving an artificial task:

$$\varphi = \sum_{i=1}^{m} p_i \to \min$$

$$\sum_{j=1}^{n}(a_{ij} \cdot x_j + p_i) = b_i, \quad \text{for } i = 1, \dots, m$$

$$x_j \geq 0, \quad \text{for } j = 1, 2, \dots, n$$
$$p_i \geq 0, \quad \text{for } i = 1, 2, \dots, m.$$

Remark 4.2. It is sufficient to add artificial variables p_i only to those constraints, where basis vectors lack.

Theorem 4.5. Artificial LPP has always an optimum.

Theorem 4.6. If the optimal solution of artificial LP is $\varphi^{\text{opt}} \neq 0$ then the original LP is infeasible i. e. it has no feasible solution.

Second phase: If $\varphi^{\text{opt}} = 0$ is true in the optimal solution of the LPP, so there are two possibilities:

1. There is no base i. e. in the optimal base remains an artificial variable:
   - find a positive number in the row corresponding with artificial variable, mark it as a pivot, and we recalculate the table with respect to the pivot.
   - if it is not possible to find a pivot in the row containing artificial variable then rows are linearly dependent. Row containing artificial variables will be left out.
   - enter the original objective function into in the row of the relative price coefficients and to continue by the 2nd step.

2. There is a base i. e. the optimal base has no artificial variable:
   - we have a basic feasible solution of the original LPP. Replace the artificial objective function by the original objective function in the row of the relative price coefficients.
   - leave out artificial columns and continue with the simplex method further.

4.3 Procedure Simplex

Suppose that T_k is the simplex table in the k-th iteration of the simplex algorithm.

```
begin
  T := T_k
  optimum := false
  unbounded := false
  while (optimum = false and unbounded = false) do
    if (c^R ≧ 0)  then optimum := true
                  else  choose any j such that c_j^R > 0
                        if (x_j ≦ 0)  then unbounded := true
                                      else  find θ = min_{i=1,...,m} { x_i0/x_ij ; where  x_ij > 0 } = x_r0/x_rj
                                            pivot is x_rj
                                            pivoting the simplex table T with respect to
                                            the pivot x_rj
                                            create a simplex table T^new after pivoting
                        end if
    end if
  end while
  T_{k+1} := T^new
end
```

4.4 Solved Examples

Example 4.1. Using the simplex method solve the following task:

$$
\begin{aligned}
15x_1 + 10x_2 &\to \max \\
2x_1 + 4x_2 &\leq 12 \\
4x_1 + 2x_2 &\leq 16 \\
2x_1 + 2 &\geq 2x_2 \\
2x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Solution:
We rewrite the given problem of LP to the standard form in order to fill the simplex table.

$$
\begin{aligned}
-15x_1 - 10x_2 &\to \min \\
2x_1 + 4x_2 + s_1 &= 12 \\
4x_1 + 2x_2 + s_2 &= 16 \\
-2x_1 + 2x_2 + s_3 &= 2 \\
2x_2 + s_4 &= 4 \\
x_{1-2},\ s_{1-4} &\geq 0
\end{aligned}
$$

LPP in the standard form has four constraints and six variables. We fill the simplex table with 6 rows and 8 columns.

Table 4.2: Simplex method – Initial table

B	x_0	x_1	x_2	s_1	s_2	s_3	s_4
—	0	-15	-10	0	0	0	0
s_1	12	2	4	1	0	0	0
s_2	16	4	2	0	1	0	0
s_3	2	-2	2	0	0	1	0
s_4	4	0	2	0	0	0	1

Columns s_1, s_2, s_3 and s_4 (slack variables) are the basis columns and we can see them as a unit submatrix of the type 4×4 in the table. The zero column consists of right sides, which must be non-negative, because the simplex table must be primarily feasible. Zero row corresponds to the relative prices, while relative prices must be zero in the basis columns. If the simplex table satisfies all these conditions, then this table is prepared to run the simplex algorithm.

According to the algorithm, we need to find a pivot. We must look for the columns with a negative relative price in the zero row. There are columns x_1 and x_2 in the Table 4.2. We select the column x_2 and we will calculate all ratios of values in the zero column and in the column x_2 for all positive values which are in the column x_2. We choose the minimum of them, i. e. $\min\{\frac{12}{4}, \frac{16}{2}, \frac{2}{2}, \frac{4}{2}\} = 1$. It is the value in the third row. Value x_{32} is the pivot, it means that the column s_3 leaves the base and the column x_2 enters into the base. We recalculate Table 4.2 by the given pivot x_{32} and we obtain a new Simplex table see Table 4.3.

Table 4.3: Simplex method – First iteration

B	x_0	x_1	x_2	s_1	s_2	s_3	s_4
—	10	-25	0	0	0	5	0
s_1	8	6	0	1	0	-2	0
s_2	14	6	0	0	1	-1	0
x_2	1	-1	1	0	0	$\frac{1}{2}$	0
s_4	2	2	0	0	0	-1	1

We got a table in which the first column has the negative relative price. It means that the table is not optimal yet, and we determine a new pivot in this column. After calculating minimum we find that an element x_{42} is the pivot. We are pivoting the table and we get a new Table 4.4:

Table 4.4: Simplex method – Second iteration

B	x_0	x_1	x_2	s_1	s_2	s_3	s_4
—	35	0	0	0	0	$-\frac{15}{2}$	$\frac{25}{2}$
s_1	2	0	0	1	0	1	-3
s_2	8	0	0	0	1	2	-3
x_2	2	0	1	0	0	0	$\frac{1}{2}$
x_1	1	1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$

We have negative relative price (in the fifth column) in this table again. By ratio criterion we find a minimum in the fifth column and we determine the pivot (element x_{15}). We use pivot operation and we get the Table 4.5:

Table 4.5: Simplex method – Third iteration

B	x_0	x_1	x_2	s_1	s_2	s_3	s_4
—	50	0	0	$\frac{15}{2}$	0	0	-10
s_3	2	0	0	1	0	1	-3
s_2	4	0	0	-2	1	0	3
x_2	2	0	1	0	0	0	$\frac{1}{2}$
x_1	2	1	0	$\frac{1}{2}$	0	0	-1

We check zero row in the Table 4.5 and we can see that there is the negative relative price in the sixth column. We determine the pivot (this is an element x_{26}). After the pivoting we are getting the Table 4.6:

Table 4.6: Simplex method – Optimal table

B	x_0	x_1	x_2	s_1	s_2	s_3	s_4
—	$\frac{190}{3}$	0	0	$\frac{5}{6}$	$\frac{10}{3}$	0	0
s_3	6	0	0	-1	1	1	0
s_4	$\frac{4}{3}$	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	1
x_2	$\frac{4}{3}$	0	1	$\frac{1}{3}$	$-\frac{1}{6}$	0	0
x_1	$\frac{10}{3}$	1	0	$-\frac{1}{6}$	$\frac{1}{3}$	0	0

In this new table there is not a negative relative price in the zero row, so it is the optimal simplex table (see Table 4.6) and we can write the optimal solution of our problem as: $\boldsymbol{x}^{\text{opt}} = (\frac{10}{3}, \frac{4}{3})^\top$. The value of the objective function is $f^{\text{opt}} = -\frac{190}{3}$. The optimal value of the original objective function is $f^{\text{opt}} = \frac{190}{3}$. $\surd$

Example 4.2. Using the simplex method solve the following task:

$$x_1 + 2x_2 - x_3 - 2x_4 + x_5 - x_6 \to \min$$
$$x_1 + x_2 - x_3 + x_4 + x_5 = 4$$
$$x_1 - x_2 + 2x_3 - x_4 + x_6 = 3$$
$$x_{1-6} \geqq 0$$

Solution:

Table 4.7: Simplex method – The filled simplex table

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6
—	0	1	2	-1	-2	1	-1
x_5	4	1	1	-1	1	1	0
x_6	3	1	-1	2	-1	0	1

The LPP is in a standard form, there are 2 constraints and 6 variables. We can fill the simplex table with 4 rows and 8 columns.

The identity submatrix is composed of columns x_5 and x_6, but the relative prices of these columns are not zero. Therefore, we must first modify the simplex table such that there were zero relative prices. Then the simplex table will be ready to run an algorithm that finds an optimal solution, if any, see Table 4.8.

Table 4.8: Simplex method – Initial table

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6
—	-1	1	0	2	-4	0	0
x_5	4	1	1	-1	1	1	0
x_6	3	1	-1	2	-1	0	1

We have only one negative relative price (-4) in the zero row and in this column we look for the pivot. There is only one positive number 1, it is the pivot. We recalculate the table with respect to that pivot and we get a new simplex table.

Table 4.9: Simplex method – First iteration

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6
—	15	5	4	-2	0	4	0
x_4	4	1	1	-1	1	1	0
x_6	7	2	0	1	0	1	1

The next pivot can be found in column x_3 and again there is the only one positive number in this column, it is 1. We recalculate the table with respect to that pivot and we get a new simplex table, see Table 4.10.

Table 4.10: Simplex method – Optimal simplex table

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6
—	29	9	4	0	0	6	2
x_4	11	3	1	0	1	2	1
x_3	7	2	0	1	0	1	1

This simplex table is optimal, because in the zero row there are not negative relative prices. The optimal solution of our problem is $\boldsymbol{x}^{\text{opt}} = (0, 0, 7, 11, 0, 0)^\top$ and the value of the objective function is $f^{\text{opt}} = -29$. $\surd$

Example 4.3. Using the simplex method solve the following task:

$$x_1 - x_2 + x_3 - 3x_4 + x_5 - x_6 - 3x_7 \to \min$$

$$\begin{aligned} 3x_3 + x_5 + x_6 &= 6 \\ x_2 + 2x_3 - x_4 &= 10 \\ -x_1 + x_6 &= 0 \\ x_3 + x_6 + x_7 &= 6 \\ x_{1-7} &\geq 0 \end{aligned}$$

Solution:
Our LPP is in standard form with 4 constraints and 7 variables. The simplex table has 6 rows and 9 columns, see Table 4.11.

Table 4.11: Simplex method – Filled in the simplex table

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
—	0	1	−1	1	−3	1	−1	−3
	6	0	0	3	0	1	1	0
	10	0	1	2	−1	0	0	0
	0	−1	0	0	0	0	1	0
	6	0	0	1	0	0	1	1

This table does not include the identity submatrix, therefore it is unable to run the simplex algorithm. The column x_1 could replace the missing column of identity submatrix,

Table 4.12: Simplex method – Modified table

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
—	0	1	−1	1	−3	1	−1	−3
x_5	6	0	0	3	0	1	1	0
x_2	10	0	1	2	−1	0	0	0
x_1	0	1	0	0	0	0	−1	0
x_7	6	0	0	1	0	0	1	1

but the third position is -1, instead of 1. It can be modify by multiplying the third row by (-1), while the simplex table will remain primarily feasible.

After this modification we already have an identity submatrix in the simplex table, which consists of columns x_5, x_2, x_1 and x_7, but the relative prices of these columns are not zero. We modify the table so that there were zero relative prices, see Table 4.13.

Table 4.13: Simplex method – Initial table

B	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
—	22	0	0	3	−4	0	2	0
x_5	6	0	0	3	0	1	1	0
x_2	10	0	1	2	−1	0	0	0
x_1	0	1	0	0	0	0	−1	0
x_7	6	0	0	1	0	0	1	1

In the zero row is only one negative relative price. In column x_4 with negative relative price we can not find pivot, because all values in this column are non positive. Therefore simplex algorithm ends and the outcome is that the task LP is indeed feasible, but unbounded. $\surd$

Example 4.4. Using the simplex method solve the following task:

$$\begin{aligned} -2x_1 - x_2 + x_3 &\to \max \\ x_1 - x_2 + x_3 &= 2 \\ -2x_1 + x_2 + x_3 &= 4 \\ x_{1-3} &\geq 0 \end{aligned}$$

Solution:
We have to rewrite LPP in a standard form.

$$\begin{aligned} 2x_1 + x_2 - x_3 &\to \min \\ x_1 - x_2 + x_3 &= 2 \\ -2x_1 + x_2 + x_3 &= 4 \\ x_{1-3} &\geq 0 \end{aligned}$$

The LPP is in the standard form with 2 constraints and 3 variables. We fill in the simplex table, which has 4 rows and 5 columns, see Table 4.14.

Table 4.14: Simplex method – Filled in the simplex table

B	x_0	x_1	x_2	x_3
—	0	2	1	−1
	2	1	−1	1
	4	−2	1	1

There is not an identity submatrix in this simplex table and we do not know how to get it by any simple modification. Therefore, we must first solve the artificial task by which we determine a basis columns. We need to add two artificial variables p_1 and p_2. Artificial LPP has the form:

$$\begin{aligned} p_1 + p_2 &\to \min \\ x_1 - x_2 + x_3 + p_1 &= 2 \\ -2x_1 + x_2 + x_3 + p_2 &= 4 \\ x_{1-3}, p_{1-2} &\geq 0 \end{aligned}$$

The Simplex table of artificial LPP in the standard form, see Table 4.15.

Table 4.15: Simplex method – Artificial LPP

B	x_0	x_1	x_2	x_3	p_1	p_2
—	0	0	0	0	1	1
p_1	2	1	−1	1	1	0
p_2	4	−2	1	1	0	1

Table 4.16: Simplex method – Artificial LPP

B	x_0	x_1	x_2	x_3	p_1	p_2
—	-6	1	0	-2	0	0
p_1	2	1	-1	1	1	0
p_2	4	-2	1	1	0	1

This artificial LPP is solved by the same simplex algorithm as in Example 4.2. Firstly we need to create a zero relative prices on the columns of identity submatrix.

The relative price is negative in the column x_3. We calculate $\min\{\frac{2}{1}; \frac{4}{1}\} = 2$. The variable p_1 leaves the base and the variable x_3 enters into the base. We use pivot operation for the table and we get new table:

Table 4.17: Simplex method – Artificial LPP

B	x_0	x_1	x_2	x_3	p_1	p_2
—	-2	3	-2	0	2	0
x_3	2	1	-1	1	1	0
p_2	2	-3	2	0	-1	1

We are looking for the negative relative price in the zero row. The relative price is negative in the column x_2. In this column there is only one positive value, so it is clearly the pivot. The variable p_2 leaves from the base and x_2 enters into the base. We can pivot this table and we get a new simplex table, see Table 4.18.

Table 4.18: Simplex method – Artificial LPP

B	x_0	x_1	x_2	x_3	p_1	p_2
—	0	0	0	0	1	1
x_3	3	$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$
x_2	1	$-\frac{3}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$

We got the optimal table. The artificial variables are not in the base and the value of objective function is 0. Simultaneously we also have an identity submatrix of the table without the last two columns, which correspond to the artificial variables. It means that we finished artificial task of LPP and we begin to solve our original task. We create a new simplex table which does not contain the last two columns of artificial variables and zero row will include the coefficients of the objective function in standard form.

Table 4.19: Simplex method – Second phase

B	x_0	x_1	x_2	x_3
—	0	2	1	-1
x_3	3	$-\frac{1}{2}$	0	1
x_2	1	$-\frac{3}{2}$	1	0

Table 4.19 has an identity submatrix, which is composed of the columns x_3 and x_2. We modify this simplex table so that we have a zero relative prices of these columns, see Table 4.20.

Table 4.20: Simplex method – Second phase – Initial step

B	x_0	x_1	x_2	x_3
—	2	3	0	0
x_3	3	$-\frac{1}{2}$	0	1
x_2	1	$-\frac{3}{2}$	1	0

We have got the optimal Simplex table, because there are no negative relative prices in the zero row. The optimal solution of our problem is $\boldsymbol{x}^{\text{opt}} = (0, 1, 3)^\top$ and the value of the objective function is $f^{\text{opt}} = -2$. $\surd$

Example 4.5. Using the simplex method solve the following task:

$$\begin{aligned} -x_1 + 2x_2 - 3x_3 &\to \max \\ -2x_1 + x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 4x_3 &= 1 \\ x_{1-3} &\geq 0 \end{aligned}$$

We write the standard form of given LPP.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &\to \min \\ -2x_1 + x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 4x_3 &= 1 \\ x_{1-3} &\geq 0 \end{aligned}$$

The LPP is in the standard form with 2 constraints and 3 variables. We fill in the simplex table, which has 4 rows and 5 columns, see Table 4.21.

Table 4.21: Simplex method – Initial table

B	x_0	x_1	x_2	x_3
—	0	1	−2	3
	2	−2	1	3
	1	2	3	4

Similarly as in Example 4.5 in the simplex table there is not an identity submatrix. First we create artificial task which determines us a basis columns. In our case, we add two artificial variable p_1 and p_2. Artificial task of LPP has the form:

$$\begin{aligned} p_1 + p_2 &\to \min \\ -2x_1 + x_2 + 3x_3 + p_1 &= 2 \\ 2x_1 + 3x_2 + 4x_3 + p_2 &= 1 \\ x_{1-3}, p_1, p_2 &\geq 0 \end{aligned}$$

The simplex table is in the form:

Table 4.22: Simplex method – Artificial task

B	x_0	x_1	x_2	x_3	p_1	p_2
—	0	0	0	0	1	1
p_1	2	−2	1	3	1	0
p_2	1	2	3	4	0	1

This artificial task is solved by the same simplex algorithm as in the previous example. First, we need to create a zero relative prices over the identity submatrix:

In the zero row, we have two negative relative prices. We select a column x_3 and determine the pivot. We calculate $\min\{\frac{2}{3}, \frac{1}{4}\} = \frac{1}{4}$.

Table 4.23: Simplex method – Artificial task

B	x_0	x_1	x_2	x_3	p_1	p_2
—	-3	0	-4	-7	0	0
p_1	2	-2	1	3	1	0
p_2	1	2	3	4	0	1

Table 4.24: Simplex method – Artificial task

B	x_0	x_1	x_2	x_3	p_1	p_2
—	$-\frac{5}{4}$	$\frac{7}{2}$	$\frac{5}{4}$	0	0	$\frac{7}{4}$
p_1	$\frac{5}{4}$	$-\frac{7}{2}$	$-\frac{5}{4}$	0	1	$-\frac{3}{4}$
x_3	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	0	$\frac{1}{4}$

We got the optimal table of the artificial task in which one artificial variable is not in the base, but the second artificial variable p_1 remained in the base i. e. $\boldsymbol{x}^{\text{art}} = (0, 0, \frac{5}{4}, 0)^T$ and the value of the objective function is $f^{\text{art}} = -\frac{5}{4} \neq 0$. This means that the original LPP have not a basis feasible solution, i. e given LPP is infeasible.

4.5 Exercises

4.1. Using the simplex method, find solutions of following linear programming problems:

a)

$$\begin{aligned} -x_1 + x_2 + x_3 + 2x_4 &\to \max \\ x_1 + 10x_3 - 4x_4 &= 25 \\ x_2 + 2x_3 + 3x_4 &= 26 \\ x_{1,2,3,4} &\geq 0 \end{aligned}$$

b)

$$\begin{aligned} -x_1 + 3x_2 &\to \max \\ 2x_1 + x_2 &\geq 6 \\ x_1 + 2x_2 &\geq 6 \\ 4x_1 - x_2 + x_3 &= 15 \\ x_{1,2,3} &\geq 0 \end{aligned}$$

c)

$$\begin{aligned} x_1 + 2x_2 &\to \max \\ 10x_1 - 4x_2 &\leq 25 \\ -2x_1 + 3x_2 &\leq 6 \\ x_{1,2} &\geq 0 \end{aligned}$$

d)

$$\begin{aligned} 7x_1 - 42x_2 &\to \min \\ 3x_1 + 5x_2 &\leq 15 \\ x_1 + x_2 &\geq 6 \\ x_{1,2} &\geq 0 \end{aligned}$$

e)

$$\begin{aligned} -3x_1 - x_2 + x_3 + 2x_4 &\to \min \\ 2x_1 - x_3 - x_4 &= 0 \\ x_1 + x_2 &= 10 \\ x_1 + 3x_3 &= 4 \\ x_{1,2,3,4} &\geq 0 \end{aligned}$$

f)

$$
\begin{aligned}
3x_1 + 2x_2 + 4x_3 &\to \max \\
x_1 + x_2 + 2x_3 &\le 4 \\
2x_1 + x_3 &\le 5 \\
2x_1 + x_2 + 3x_3 &\le 7 \\
x_{1,2,3} &\ge 0
\end{aligned}
$$

g)

$$
\begin{aligned}
2x_1 + 3x_2 + 3x_3 &\to \max \\
3x_1 + 2x_2 + x_4 &= 60 \\
x_1 - x_2 - 4x_3 &\ge -10 \\
2x_1 - 2x_2 + 5x_3 &\le 50 \\
x_{1,2,3,4} &\ge 0
\end{aligned}
$$

h)

$$
\begin{aligned}
x_1 - 2x_2 - 3x_3 - x_4 &\to \max \\
x_1 - x_2 - 2x_3 - x_4 &\le 4 \\
2x_1 + x_3 - 4x_4 &\le 2 \\
-2x_1 + x_2 + x_4 &\le 1 \\
x_{1,2,3,4} &\ge 0
\end{aligned}
$$

4.2. Solve the linear programming problem which is given in the example 2.3.

4.3. Solve the linear programming problem which is given in the example 2.4.

4.4. Solve the linear programming problem which is given in the example 2.5.

4.5. Solve the linear programming problem which is given in the example 2.6.

4.6. Solve the linear programming problem which is given in the example 2.8.

4.7. Solve the linear programming problem which is given in the example 2.9.

4.6 Solutions

4.1 a) $\boldsymbol{x}^{opt} = (0; 21; 5/2; 0)^\top,\ f(\boldsymbol{x})^{opt} = 47/2$

b) The LPP is faesible but unbounded.

c) $\boldsymbol{x}^{opt} = (9/2; 5)^\top,\ f(\boldsymbol{x})^{opt} = 29/2$

d) The LPP is unfeasible.

e) $\boldsymbol{x}^{opt} = (4/7; 66/7; 8/7; 0)^\top,\ f(\boldsymbol{x})^{opt} = -10$

f) $\boldsymbol{x}^{opt} = (5/2; 3/2; 0)^\top,\ f(\boldsymbol{x})^{opt} = 21/2$

g) $\boldsymbol{x}^{opt} = (8; 18; 0; 0)^\top,\ f(\boldsymbol{x})^{opt} = 70$

h) $\boldsymbol{x}^{opt} = (7; 0; 0; 3)^\top,\ f(\boldsymbol{x})^{opt} = 4$

4.2 MBF: \$ 7500; CD: \$ 2500; H-RF: \$ 2000. Profit is \$ 965 per year.

4.3 This LPP hasn't any feasible solution.

4.4 The farmer reaches the biggest profit \$ 3200 if: 4 hectares for wheat; 4 hectares for rye.

4.5 Source A: 2 tons; source B: 4 tons. The biggest daily yield of gold is 16 oz.

4.6 Carpenter's plan: 4/3 tables, 32/3 chairs. The biggest profit is £440/3.

4.7 The company reaches the biggest daily profit \$ 650 if it produces 100 scientific calculators and 170 graphing calculators daily.

Chapter 5

Dual Simplex Method

5.1 Dual Simplex Method – Algorithm

The dual algorithm of the simplex method is used to solve the primary tasks of the linear programming problem. However, while the primary simplex algorithm must have the primary feasible table, the dual algorithm we use, if the table is not primarily feasible (the primary algorithm cannot be used), but the table is dual feasible. The dual algorithm is compared with the primary algorithm like the primary a little bit modified. The coefficients $\boldsymbol{c}$ of the objective function and the right sides $\boldsymbol{b}$ have an inverse role. There, we also move from one basis feasible solution to another, but we try to maintain the dual feasibility. The pivot is choosen by another way:

- Choose the pivot in the i-th row, where the value $x_{i0} < 0$.
- For all $x_{ij} < 0$ calculate $\frac{x_{0j}}{x_{ij}}$ in the i-th row and we determine the λ.

$$\lambda = \frac{x_{0k}}{x_{ik}} = \max\left\{\frac{x_{0j}}{x_{ij}};\ \text{for } j \text{ such that } x_{ij} < 0\right\}.$$

- Thus determined x_{ik} is a pivot and the table is pivoted by the same way as in the primary simplex algorithm.

We describe in the table 5.1 on the page 88 how to determining the pivot in the primary and the dual simplex method algorithm.

Table 5.1: Determining the pivot in the primary and the dual simplex method algorithm.

Primary algorithm of the SM	Dual algorithm of the SM
choose j-th column to the base so that $x_{0j} < 0$	choose i-th row out of the base so that $x_{i0} < 0$
calculate $\frac{x_{i0}}{x_{ij}}$, $\forall x_{ij} > 0$ in j-th column	calculate $\frac{x_{0j}}{x_{ij}}$, $\forall x_{ij} < 0$ in i-th row
$\frac{x_{k0}}{x_{kj}} = \min\left\{\frac{x_{i0}}{x_{ij}};\text{ for } i \text{ such that } x_{ij} > 0\right\}$	$\frac{x_{0k}}{x_{ik}} = \max\left\{\frac{x_{0j}}{x_{ij}};\text{ for } j \text{ such that } x_{ij} < 0\right\}$
pivot is the element x_{kj} (must be positive)	pivot is the element x_{ik} (must be negative)
if in the each column, where $x_{0j} < 0$, is each $x_{ij} \leq 0$, then LPP is unbounded	if in the each column, where $x_{i0} < 0$, is each $x_{ij} \geq 0$, then LPP is unfeasible

5.2 Procedure Dual Simplex

Suppose that T_k is the simplex table in the k-th iteration of the simplex algorithm.

```
begin
    T := T_k
    optimum := false
    unbounded := false
    while (optimum = false and unbounded = false) do
        if (x_i0 ≧ 0)  then optimum := true
                       else  choose any i such that x_i0 < 0
                             if (∀j x_ij ≧ 0)   then unbounded := true
                                                else  find
                                                      λ = x_0k / x_ik = max { x_0j / x_ij ; for j such that x_ij < 0 }
                                                      pivot is x_ik
                                                      pivoting the simplex table T with respect to
                                                      the pivot x_ik
                                                      create a simplex table T^new after pivoting
                             end if
        end if
    end while
    T_{k+1} := T^new
end
```

5.3 Solved Examples

Example 5.1. Solve the following problem using the simplex method:

$$\begin{aligned} 3x_1 + 2x_2 + 3x_3 &\to \min \\ x_1 - x_2 - x_3 &\geq 2 \\ x_1 + x_2 + x_3 &\geq 4 \\ x_1 - 2x_2 + x_3 &\geq 1 \\ x_{1-3} &\geq 0 \end{aligned}$$

Solution:
The LPP is converted into a standard form.

$$\begin{aligned} 3x_1 + 2x_2 + 3x_3 &\to \min \\ x_1 - x_2 - x_3 - s_1 &= 2 \\ x_1 + x_2 + x_3 - s_2 &= 4 \\ x_1 - 2x_2 + x_3 - s_3 &= 1 \\ x_{1-3}, s_{1-3} &\geq 0 \end{aligned}$$

The LPP in standard form has three constrains and six variables. We are fill the simplex table with five rows and eight columns.

Table 5.2: Dual simplex method – First step.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	0	3	2	3	0	0	0
	2	1	-1	-1	-1	0	0
	4	1	1	1	0	-1	0
	1	1	-2	1	0	0	-1

In this table we do not have the unit submatrix. We can multiply each row by the number (-1) and we receive the unit submatrix, but the simplex table would be primarily infeasible. See Table 5.3
This table is dual feasible. We use the *Dual Simplex Method*. We choose a negative value in the zero row. If we find the pivot in this row, then the variable corresponding to the choosen row goes out of the base and variable in which column we found the pivot goes to the base. We choose the last row and we have to determine the pivot in this row. We calculate $\max\{\frac{3}{-1}; \frac{3}{-1}\} = -3$. Let us choose a pivot in the first column x_1. We pivotal the table with respect to the specified pivot and we get a new Simplex table, which is still

Table 5.3: Dual simplex method – Second step.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	0	3	2	3	0	0	0
s_1	-2	-1	1	1	1	0	0
s_2	-4	-1	-1	-1	0	1	0
s_3	-1	-1	2	-1	0	0	1

Table 5.4: Dual simplex method – Third step.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	-3	0	8	0	0	0	3
s_1	-1	0	-1	2	1	0	-1
s_2	-3	0	-4	0	0	1	-1
x_1	1	1	-2	1	0	0	-1

primarily unfeasible, but the dual feasible - Table 5.4. We again select the pivot under the dual simplex method.
We find a negative value in the zero column and in this row we determine the pivot. Let it be the second row, in which we determine the pivot. We calculate $\max\{\frac{8}{-4}; \frac{3}{-1}\} = -2$. The element x_2 enter to the base and s_2 goes out from the base. We recalculate the table with respect to the specified pivot, see Table 5.5.

Table 5.5: Dual simplex method – Fourth step.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	-9	0	0	0	0	2	1
s_1	$-\frac{1}{4}$	0	0	2	1	$-\frac{1}{4}$	$-\frac{3}{4}$
x_2	$\frac{3}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{1}{4}$
x_1	$\frac{5}{2}$	1	0	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$

We have only one negative value in the zero column. We determine the pivot in the first row. Calculate $\max\{\frac{2}{-\frac{1}{4}}, \frac{1}{-\frac{3}{4}}\} = -\frac{4}{3}$. The variable s_3 goes to the base and variable s_1 goes out from the base.

Table 5.6: Dual simplex method – Fifth step.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	$-\frac{28}{3}$	0	0	$\frac{8}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	0
s_1	$\frac{1}{3}$	0	0	$-\frac{8}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	1
x_2	$\frac{2}{3}$	0	1	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	0
x_1	$\frac{8}{3}$	1	0	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	0

We obtained an optimal Table 5.6 with an optimal solution of LPP: $\boldsymbol{x}^{opt} = (\frac{8}{3}, \frac{2}{3}, 0)^\top$ and the value of the objective function is $f^{opt} = \frac{28}{3}$. $\surd$

5.4 Exercises

5.1. Using the dual simplex method, find solutions of the following linear programming problems:

a)

$$\begin{aligned} 6x_1 + 4x_2 + 7x_3 &\to \min \\ x_1 + 3x_3 &\geq 5 \\ 3x_1 + x_2 + x_3 &\geq 2 \\ -x_1 + x_2 &\geq 1 \\ x_{1,2,3} &\geq 0 \end{aligned}$$

b)

$$\begin{aligned} -x_1 - 2x_2 - x_3 &\to \max \\ -2x_1 + 3x_3 &\geq -1 \\ 2x_1 - x_2 + x_3 &\geq 1 \\ 3x_1 + 2x_2 - x_3 &\geq 0 \\ x_{1,2,3} &\geq 0 \end{aligned}$$

5.2. In order to ensure optimal health, a lab technician needs to feed rabbits a daily diet containing a minimum of 24 g of fat, 36 g of carbohydrates, and 4 g of protein. But the rabbits should be fed no more than five ounces of food a day. Rather than order rabbit food that is custom-blended, it is cheaper to order Food X and Food Y, and blend them for an optimal mix. Food X contains 8 g of fat, 12 g of carbohydrates, and 2 g of protein per ounce, and costs $0,20$ \$ per ounce. Food Y contains 12 g of fat, 12 g of carbohydrates, and 1 g of protein per ounce, at a cost of $0,30$ \$ per ounce.

a) What is the optimal blend?

b) Solve this problem, if food X contains 8 g of fat, 6 g of carbohydrates, and 2 g of protein per ounce.

5.5 Solutions

5.1 a) $\boldsymbol{x}^{opt} = (0; 1; 5/3)^\top,\ f(\boldsymbol{x})^{opt} = 47/3$

b) $\boldsymbol{x}^{opt} = (1/2; 0; 0)^\top,\ f(\boldsymbol{x})^{opt} = -1/2$

5.2 a) 3 ounces of X, 0 ounces of Y

b) 2/3 ounces of X, 8/3 ounces of Y

Chapter 6

Integer Linear Programing Problem

6.1 Formulation of the Integer Linear Programing Problem

Definition 6.1 (ILP Problem)**.** The linear programming problem is called the integer linear programming problem if it is in the following form:

$$f(\boldsymbol{x}) = \boldsymbol{c}^\top \cdot \boldsymbol{x} \to \min\ (\max)$$

$$\sum_{i=1}^{m} \boldsymbol{a_i} \cdot \boldsymbol{x} \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} \boldsymbol{b} \tag{6.1}$$

$$x_j \leq\geq 0; \qquad x_j \in \mathbb{Z}; \qquad j = 1, 2, \ldots, n \text{ , where}$$

coefficients of the objective function, coefficients of the right hand sides and elements of the matrix of constrains are integers.

Remark 6.1. The matrix notation of ILPP with n variables and m constrains in the standard form is as follows:

$$\begin{aligned} & f(\boldsymbol{x}) = \boldsymbol{c}^\top \cdot \boldsymbol{x} \to \min \\ & \boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b} \\ & \quad x_i \geq 0; \qquad x_j \in \mathbb{Z}; \quad \text{for } j = 1, 2, \ldots, n, \\ & \quad \boldsymbol{A} \in \mathbb{Z}^{m \times n}; \qquad \boldsymbol{c} \in \mathbb{Z}^n; \qquad \boldsymbol{b} \in \mathbb{Z}^m. \end{aligned}$$

In general, it is sufficient to require only unknown vector $\boldsymbol{x}$ to be an integer. If not all variables are required to be integer we called it as – *mixed program.*

Definition 6.2. If in the task of the integer linear programming problem (6.1) the condition that variables ($\boldsymbol{x} \in \mathbb{Z}^n$) are integer is omitted, we obtain the task of the linear programming problem, which is called the *relaxation* of ILP (6.1).

Remark 6.2. We denote:

- Feasible set of LPP as F_{LPP}
- Feasible set of ILPP as F_{ILPP}
- Sets of optimal solutions of LPP as F_{LPP}^{opt}
- Sets of optimal solutions of ILPP as F_{ILPP}^{opt}
- Optimal values of objective functions of LPP as f_{LPP}^{opt}
- Optimal values of objective functions of ILPP as f_{ILPP}^{opt}

Theorem 6.1. [Relation between ILPP and its relaxation – 1] The following holds: $F_{ILPP} \subseteq F_{LPP}$.

Theorem 6.2 (Relation between ILPP and its relaxation – 2)**.** If an optimal solution of relaxation of ILPP (6.1) is an integer, then it is an optimal solution of ILP (6.1) too.

Theorem 6.3 (Relation between ILPP and its relaxation – 3)**.** Let all the entries of the matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$ be integer. If relaxation of ILP is unbonded and F_{ILPP} is nonempty, then ILPP is unbounded too.

Theorem 6.4 (Relation between ILPP and its relaxation – 4)**.** If the relaxation of ILPP (6.1) is infeasible, then ILP (6.1) is infeasible too.

See example 6.10.

6.2 Integer Linear Programing Problem in $\mathbb{R}^2$

This subsection lists some examples of two-variable ILPs and their representatiosn in $\mathbb{R}^2$. In the following examples you graphically draw the set of feasible solutions and the optimal solution of the relaxations of ILPP, also the set of feasible solutions and the optimal solution of ILPP.

Example 6.1. Sove the following ILPP:

$$\begin{aligned} x_1 + 2x_2 &\to \max \\ 10x_1 + 7x_2 &\leq 35 \\ -2x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

Solution:
In the figure 6.1 we can see the set of feasible solutions (left figure) of the relaxation ILPP. This relaxation has one optimal solution $\boldsymbol{x}_r^{opt} = (7/8, 15/4)^\top$, which is not integer solution. The set of feasible solution of ILPP is shown on the right figure – it is the set of marked points. There is one optimal solution of ILPP: $\boldsymbol{x}^{opt} = (1, 3)^\top$.

$\surd$

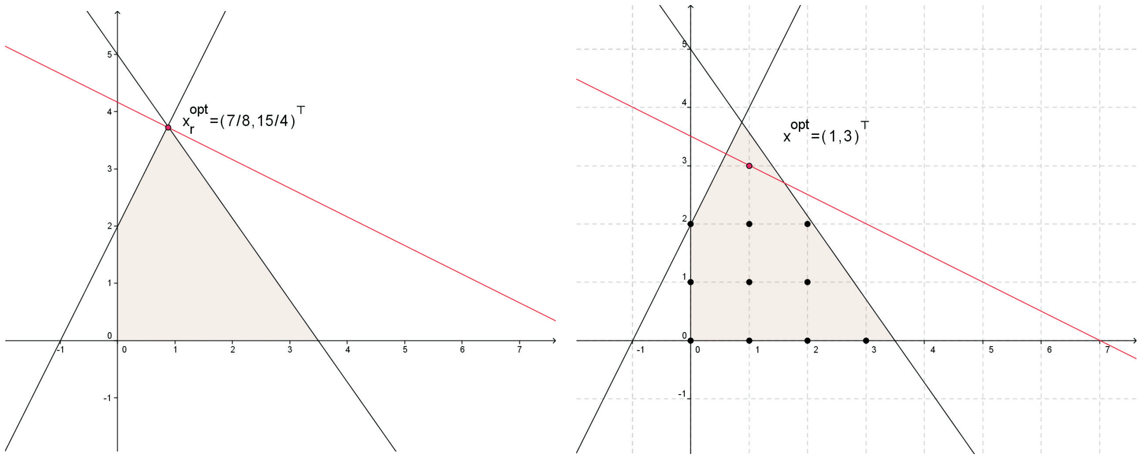

Figure 6.1: The graphical representation of the ILPP – example 6.1.

Example 6.2. Let we have ILPP:

$$
\begin{aligned}
x_1 + x_2 &\to \max \\
10x_1 + 7x_2 &\leq 35 \\
-2x_1 + x_2 &\leq 2 \\
x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}.
\end{aligned}
$$

Solution:
In the figure 6.2 we can see the set of feasible solutions (left figure) of the relaxation ILPP. This relaxation has one optimal solution $\boldsymbol{x}_r^{opt} = (7/8, 15/4)^\top$, which is not an integer solution. The set of feasible solutions of ILPP is shown on the right figure – it is the set of marked points. We can see that the given ILPP has two optimal solutions: $\boldsymbol{x}_1^{opt} = (1,3)^\top$ and $\boldsymbol{x}_2^{opt} = (2,2)^\top$.

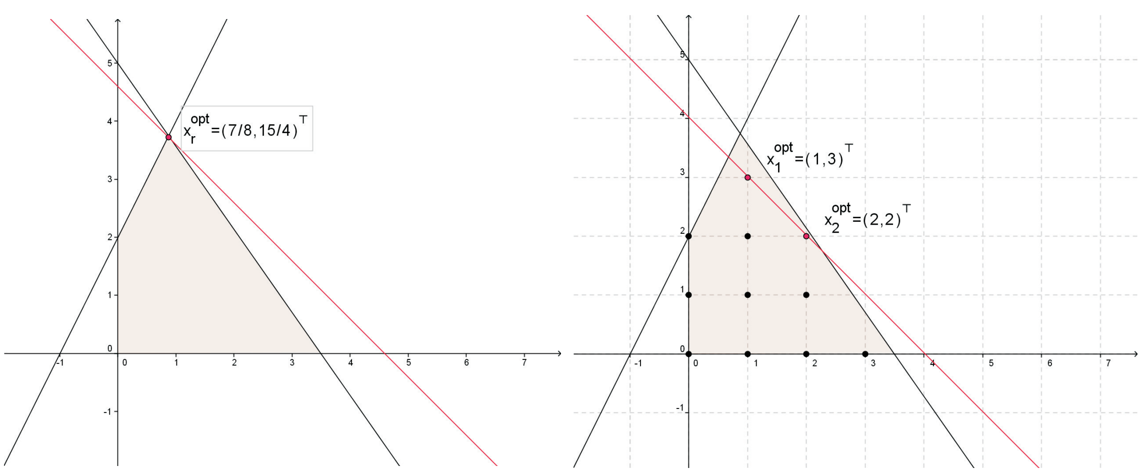

Figure 6.2: The graphical representation of the ILPP – example 6.2.

$\surd$

Example 6.3. Let us have ILPP:

$$
\begin{aligned}
x_1 + x_2 &\to \max \\
10x_1 + 8x_2 &\geq 41 \\
3x_1 + 2x_2 &\leq 12 \\
-x_1 + 2x_2 &\leq 2 \\
x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}.
\end{aligned}
$$

Solution:
In the figure 6.3 we can see the set of feasible solutions of the relaxation ILPP (left side of the figure). This relaxation has more optimal solution $\boldsymbol{x}_r^{opt} = (5/2, 9/4)^\top$, but it is not an integer solution. The set of feasible solutions of ILPP is shown on the right figure – it is the empty set. This means that the ILPP is infeasible.

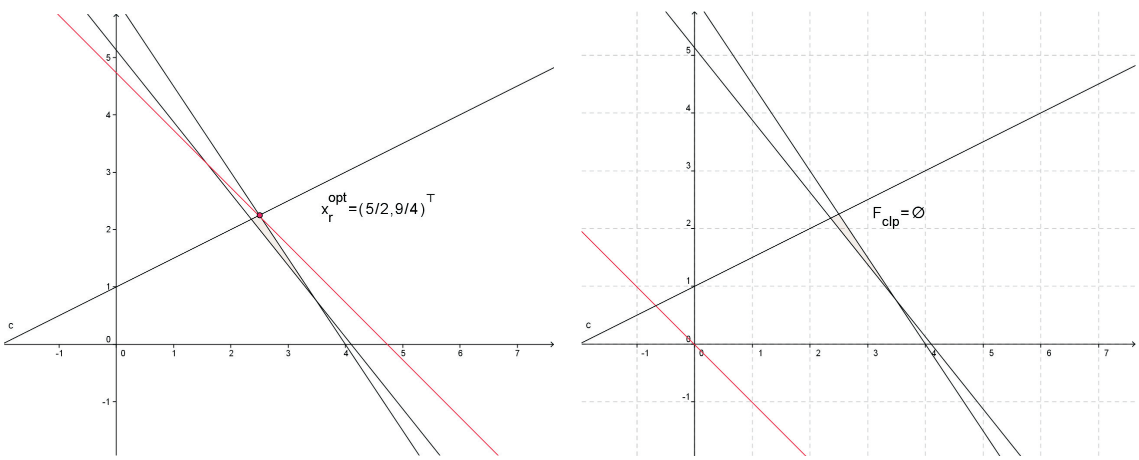

Figure 6.3: The graphical representation of the ILPP – example 6.3.

$\surd$

Example 6.4. Let us have ILPP:

$$
\begin{aligned}
x_1 - x_2 &\to \max \\
7x_1 + 2x_2 &\geq 14 \\
4x_1 + 9x_2 &\leq 45 \\
x_1 - x_2 &\leq 3 \\
x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}.
\end{aligned}
$$

Solution:
The set of feasible solutions of relaxation of the given ILPP is drown on the right side of

figure 6.4. This relaxation has more then one optimal solutions – an infinite number of optimal solutions.The set of optimal solutions of relaxation is the line segment $\overline{BC}$. The set of feasible solutions of the given ILPP is drawing on the left side of figure 6.4. It has more then one (three) optimal solutions $\boldsymbol{x}^{opt} \in \{(3,0)^\top, (4,1)^\top, (5,2)^\top\}$.

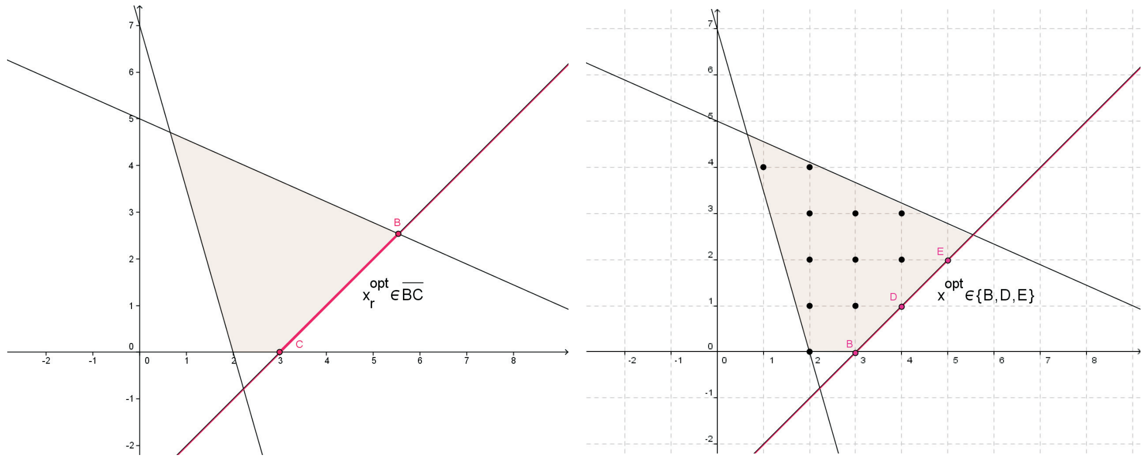

Figure 6.4: The graphical representation of the ILPP – example 6.4.

$\surd$

Example 6.5. Next ILPP is given as:

$$
\begin{aligned}
-3x_1 + x_2 &\to \min\\
7x_1 + 2x_2 &\geq 14\\
4x_1 + 9x_2 &\leq 45\\
6x_1 - 2x_2 &\leq 23\\
x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}.
\end{aligned}
$$

Solution:
The set of feasible solutions of relaxation of the given ILPP is drawing on the right side of figure 6.5. This relaxation has (as in the previous example) more than one optimal solution – an infinite number of optimal solutions and the set of optimal solutions of relaxation is the line segment $\overline{BC}$. The set of feasible solutions of the given ILPP is drawing on the left side of figure 6.5, But in this case the given ILPP has just one optimal solution $\boldsymbol{x}^{opt} = (4,1)^\top$.

$\surd$

Example 6.6. Let we have ILPP:

$$
\begin{aligned}
-6x_1 + 5x_2 &\to \max\\
14x_1 + 7x_2 &\leq 49\\
8x_1 - 11x_2 &\leq 4\\
6x_1 - 5x_2 &\geq -3\\
x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}.
\end{aligned}
$$

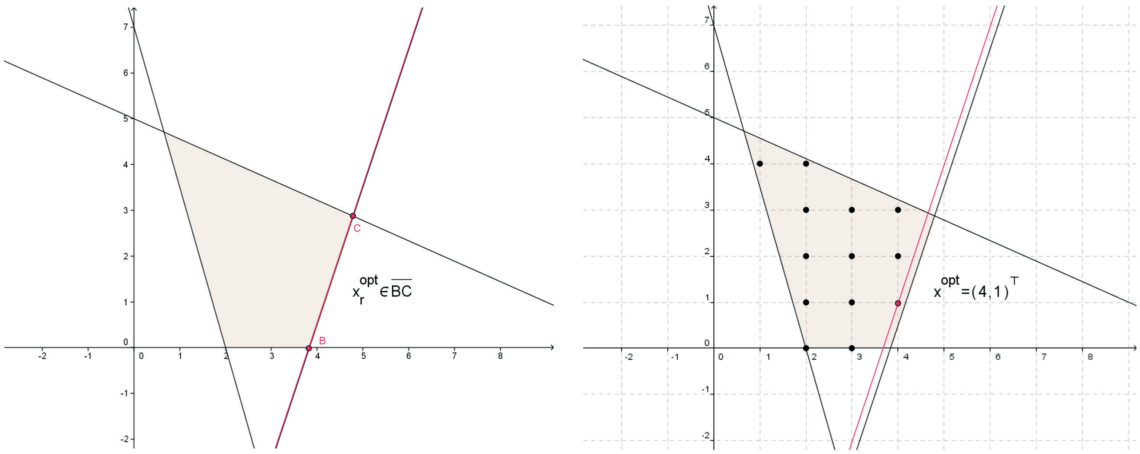

Figure 6.5: The graphical representation of the ILPP – example 6.5.

Solution:
The set of feasible solutions of relaxation of the given ILPP is drawn on the right side of the figure 6.6. This relaxation has (as in the two previous example) more then one optimal solution – the set of optimal solutions of relaxation is the line segment $\overline{FG}$. But we can see on the left side of the figure 6.6 The feasible set of ILPP is empty and ILPP is infeasible.

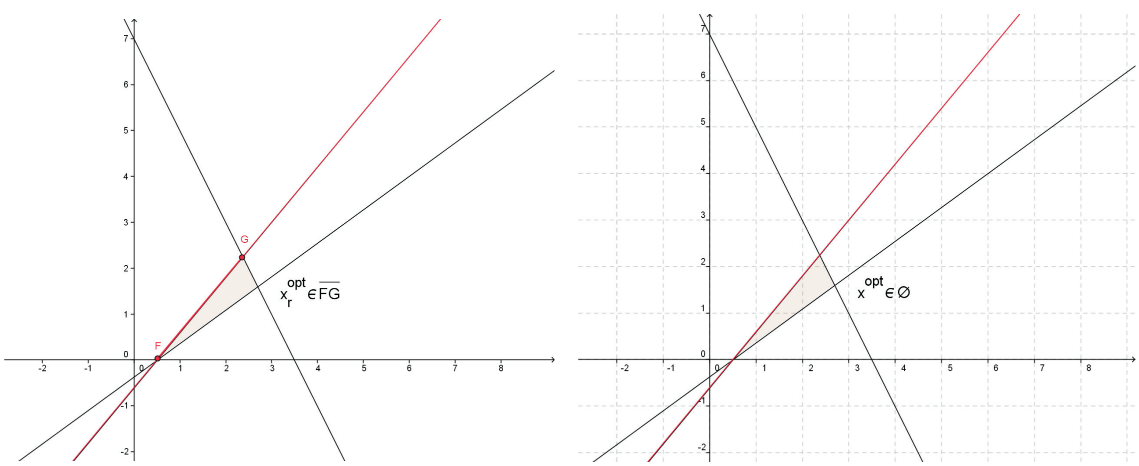

Figure 6.6: The graphical representation of the ILPP – example 6.6.

$\surd$

In the two following examples we have a unbounded feasible sets of relaxations but feasible sets of ILPP are of different types.

Example 6.7. Let we have ILPP:

$$\begin{aligned} 2x_1 + x_2 &\to \max \\ x_1 - 4x_2 &\le -3 \\ -2x_1 + x_2 &\le -1 \\ -5x_1 + 6x_2 &\ge -9 \\ x_1, x_2 &\ge 0; \quad x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

Solution:
The relaxation of the given ILPP is feasible and unbounded – see figure 6.7 – left side. The ILPP is also feasible and unbounded – see figure 6.7 – right side. $\surd$

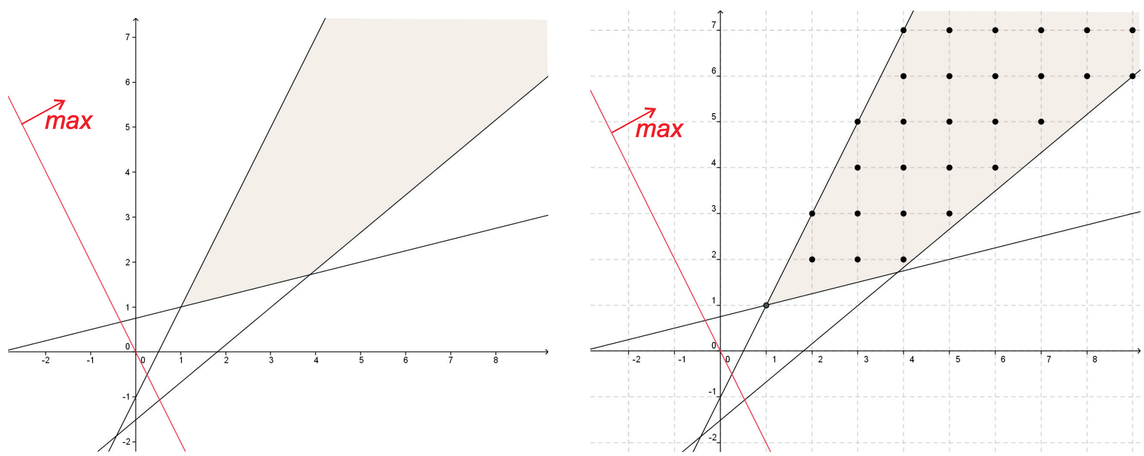

Figure 6.7: The graphical representation of the ILPP – example 6.7.

Example 6.8. Let we have ILPP:

$$\begin{aligned} 2x_1 + 3x_2 &\to \max \\ 3x_1 - 3x_2 &\ge -5 \\ 3x_1 - 3x_2 &\le -4 \\ x_1 + x_2 &\ge 3 \\ x_1, x_2 &\ge 0; \quad x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

Solution:
The relaxation of the given ILPP is feasible and unbounded – see figure 6.8 – left side, but the ILPP is infeasible – see figure 6.8 – right side. $\surd$

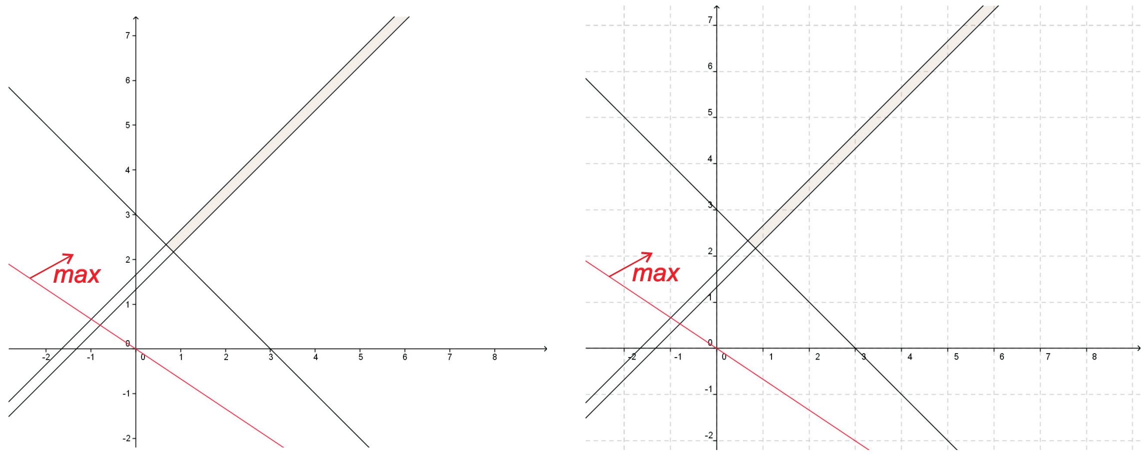

Figure 6.8: The graphical representation of the ILPP – example 6.8.

Example 6.9. Let us have ILPP:

$$\begin{aligned} 2x_1 + 3x_2 &\to \min \\ 4x_1 + 5x_2 &\leq 16 \\ 7x_1 + 4x_2 &\geq 42 \\ -2x_1 + 3x_2 &\leq -4 \\ x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

Solution:
As we can see in the figure 6.9 the relaxation of the given ILPP is infeasible. According to the Theoreme 6.1 the ILPP is infeasible too. $\surd$

Observation:
We might have noticed in the previous examples that a feasible set of relaxation of ILPP could be infeasible, feasible bounded and feasible unbounded. A feasible bounded set could have one or more than one optimal solutions. A feasible set of ILPP could be infeasible, feasible bounded and feasible unbounded. A feasible bounded set could have one or more than one optimal solutions. The next table clearly shows, which options are possible ($\surd$) or are not possible (–) for the pair "relaxation of ILPP – ILPP".

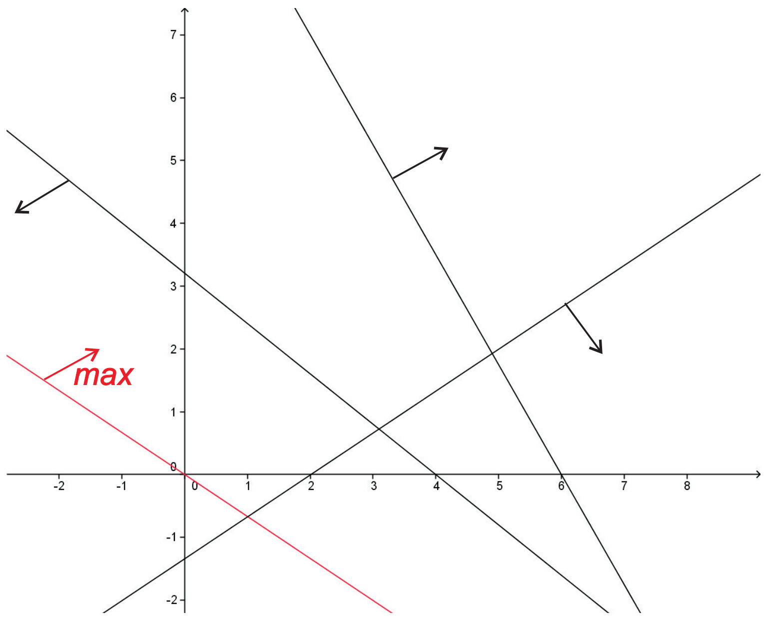

Figure 6.9: The graphical representation of the ILPP – example 6.9.

6.3 Gomory's Fractional Algorithm

We can solve tasks of the integer linear programming problem with two variables graphically with some limitations. But what happens if ILPP has more than two decision variables? In the subsection 4.1 is an example **??**, which is solved by simplex method. The relaxation of the ILP has an integer solution. This solution was also solution of ILP. If the relaxation of ILP is not integer solution, we can solve it by method of the so-called cutting hyperplane, otherwise also called *Gomory's fractional algorithm*.

First, using the simplex method we solve the ILP relaxation. Gomory fractional algorithm adds to the problems of linear programming constrains - *Gomory cuts* which narrow down the set of feasible solutions of some parts do not containing the points with integer values. For solving of the expanded task about such a cut is preferable to use the dual simplex method.

Let task of the integer linear programming problem is given in the standard form:

$$
\begin{aligned}
&f(\boldsymbol{x}) = \boldsymbol{c}^\top \cdot \boldsymbol{x} \to \min \\
&\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b} \\
&\quad x_i \geq 0; \qquad x_j \in \mathbb{Z}; \quad \text{for } j = 1, 2, \ldots, n, \\
&\quad \boldsymbol{A} \in \mathbb{Z}^{m \times n}; \qquad \boldsymbol{c} \in \mathbb{Z}^n; \qquad \boldsymbol{b} \in \mathbb{Z}^m.
\end{aligned}
$$

Let us have the optimal table for a relaxation of ILP. The elements of the optimal table will be denoted γ_{ij}.

Table 6.1: Information about a relaxation of ILPP and ILPP.

relaxation of ILPP	ILPP			
	1 optimum	more than 1 optimum	feasible unbounded	infeasible
1 optimum	$\surd$	$\surd$	–	$\surd$
more than 1 opt.	$\surd$	$\surd$	–	$\surd$
feasible unbounded	–	–	$\surd$	$\surd$
infeasible	–	–	–	$\surd$

Corollary 6.1. After the addition of Gomory cut (6.2):

$$-\sum_{j\notin B}\{\gamma_{ij}\}\cdot x_j + g = -\{\gamma_{i0}\} \tag{6.2}$$

to the optimal table (LPP) is not excluded any integer feasible point, but exclude the currently optimal solution (LPP), where γ_{i0} is not an integer. The new table is basic, primary infeasible and optimal.

Theorem 6.5 (Finality Gomory algorithm)**.** Gomory algorithm
(a) chooses the first row with non-integer y_{i0},
(b) use the lexicographic version of the dual algorithm.
If the objective function (LPP) is bounded from above, then the algorithm finds after the final number of steps the integer solution (ILPP) or finds that (ILPP) is infeasible.

6.4 Solved Examples

Example 6.10. We need to buy some filing cabinets. There are two types of them: $S40$ and $Sk60$. You know that Cabinet $S40$ costs 10 € per unit, requires 0,55 m^2 of floor space, and holds 0,22 m^3 of files. Cabinet $Sk60$ costs 20 € per unit, requires 0,74 m^2 of floor space, and holds 0,56 m^3 of files. The office has room for no more than 6,6 m^2 of cabinets. Our budget is 140 € . How many of which model should we buy, in order to maximize storage volume?
Solution:
We denote number of $S40$ as x_1 and number of $Sk60$ as x_2:

$$\begin{aligned} 0{,}22x_1 + 0{,}56x_2 &\to \max \\ 10x_1 + 20x_2 &\leq 140 \\ 0{,}55x_1 + 0{,}74x_2 &\leq 6{,}6 \\ x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

We multiply the objective function and constraints by the appropriate number in order to have integer coefficients:

$$\begin{aligned} 11x_1 + 28x_2 &\to \max \\ x_1 + 2x_2 &\leq 14 \\ 55x_1 + 74x_2 &\leq 660 \\ x_1, x_2 &\geq 0; \quad x_1, x_2 \in \mathbb{Z}. \end{aligned}$$

We obtain ILP. By omitting conditions $x_1, x_2 \in \mathbb{Z}$, we have relaxation of the given ILP and we transform it into standard form:

$$\begin{aligned} -11x_1 - 28x_2 &\to \min \\ x_1 + 2x_2 + s_1 &= 14 \\ 55x_1 + 74x_2 + s_2 &= 660 \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

This relaxation is solved with using simplex method:

B	x_0	x_1	x_2	s_1	s_2
—	0	-11	-28	0	0
s_1	14	1	2	1	0
s_2	660	55	74	0	1

The table is not optimal and we must use a pivot operation. The number 2 on the position $(1;2)$ is the pivot:

B	x_0	x_1	x_2	s_1	s_2
—	196	3	0	14	0
x_2	7	1/2	1	1/2	0
s_2	142	18	0	−37	1

The optimal solution of the relaxation of given ILP is $\boldsymbol{x}^{opt} = (0,7)^\top$; $f^{opt} = -196 : (-50) = 3,92$. Because the solution $\boldsymbol{x}^{opt} = (0,7) \in \mathbb{Z}^2$ than this solution is solution of givet ILP too. We should order 7 pieces of Cabinet $Sk60$ and we obtain $3,92\ m^3$ of storage volume. $\surd$

Example 6.11. Carpentry manufactures three types of tables. They use three different kinds of wooden boards for their production. Consumption of these boards to produce one table of various kinds, stocks boards and the selling profit of one table are given in the following table:

tables\boards	B_1	B_2	B_3	profit (€)
T_1	2	4	0	8
T_2	1	0	1	10
T_3	1	2	1	12
stocks	80	50	40	

The task is to schedule production plan so that the profit will be maximum.
Solution:
The standard form of the mathematical model of the ILPP is as follows:

$$\begin{aligned} 8x_1 + 10x_2 + 12x_3 &\to \max \\ 2x_1 + x_2 + x_3 + s_1 &= 80 \\ 4x_1 + 2x_3 + s_2 &= 50 \\ x_2 + x_3 + s_3 &= 40 \\ x_1,\ x_2,\ x_3,\ s_1,\ s_2,\ s_3 \geq 0; &\quad x_1,\ x_2,\ x_3 \in \mathbb{Z}. \end{aligned}$$

We can fill the simplex table of the relaxation of the ILPP.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	0	−4	−5	-6	0	0	0
s_1	80	2	1	1	1	0	0
s_2	50	4	0	2	0	1	0
s_3	40	0	1	1	0	0	1

The table is basis, primary feasible but it is not an optimal table. We must use the pivot operation. We recalculate the last table by the given pivot $x_{32} = 1$ and we get a new simplex table.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	200	-4	0	-1	0	0	5
s_1	40	2	0	0	1	0	-1
s_2	50	4	0	2	0	1	0
x_2	40	0	1	1	0	0	1

We recalculate the table again but the given pivot is $x_{23} = 2$ and we get a new simplex table.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	225	-2	0	0	0	1/2	5
s_1	40	2	0	0	1	0	-1
x_3	25	2	0	1	0	1/2	0
x_2	15	-2	1	0	0	-1/2	1

Since neither this table is an optimal, we use pivot operation again and the pivot is $x_{21} = 2$.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3
—	250	0	0	1	0	1	5
s_1	15	0	0	-1	1	$-1/2$	-1
x_1	25/2	1	0	1/2	0	1/4	0
x_2	40	0	1	1	0	0	1

This table is optimal and the solution of the relaxation is $x_r^{opt} = (25/2, 40, 0)^\top$. As the solution of relaxation is not an integer, it is not a solution ILPP. The value of variable x_1 is not an integer we add Gomory cut according row of the simplex table which belongs to x_1:

$$\{1\} \cdot x_1 + \{0\} \cdot x_2 + \{1/2\} \cdot x_3 + \{0\} \cdot s_1 + \{1/4\} \cdot s_2 + \{0\} \cdot s_3 - g = \{25/2\}.$$

So we have:

$$-1/2 \cdot x_3 - 1/4 \cdot s_2 + g = -1/2.$$

We add one column and one row for g to table – for Gomory cut:

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3	g
—	250	0	0	1	0	1	5	0
s_1	15	0	0	-1	1	$-1/2$	-1	0
x_1	25/2	1	0	1/2	0	1/4	0	0
x_2	40	0	1	1	0	0	1	0
g	$-1/2$	0	0	$-1/2$	0	$-1/4$	0	1

The obtained simplex table is primary infeasible, but dual feasible and basis. We can use dual simplex algorithm and pivot is $x_{43} = -1/2$.

B	x_0	x_1	x_2	x_3	s_1	s_2	s_3	g
—	249	0	0	0	0	1/2	5	2
s_1	16	0	0	0	1	0	-1	-2
x_1	12	1	0	0	0	0	0	1
x_2	39	0	1	0	0	$-1/2$	1	2
x_3	1	0	0	1	0	1/2	0	-2

One can see that table is optimal and solution is integer. So the solution of the ILPP is $\boldsymbol{x}^{opt} = (12, 39, 1)^\top$ a $f^{opt} = 249$. Carpentry will have maximal profit if it manufactures 12 tables T_1, 39 tables T_2 and 1 table T_3. The profil will be 249 €. $\surd$

6.5 Exercises

6.1. Find solutions of following integer linear programming problems:

a)

$$\begin{aligned} 2x_1 + 3x_2 &\to \max \\ 3x_1 + 2x_2 &\leq 400 \\ 1,5x_1 + x_2 &\leq 150 \\ 3x_1 + 5x_2 &\leq 300 \\ x_{1,2} &\geq 0 \\ x_{1,2} &\in \mathbb{Z} \end{aligned}$$

b)

$$\begin{aligned} 80x_1 + 114y_1 &\to \max \\ x_1 - 2x_2 &\geq 0 \\ 0,5x_1 + x_2 &\leq 19 \\ x_1 + 2x_2 &\leq 40 \\ 2x_1 + 5x_2 &\leq 15 \\ x_{1,2} &\geq 0 \\ x_{1,2} &\in \mathbb{Z} \end{aligned}$$

c)

$$\begin{aligned} x_1 + x_2 + x_3 &\to \max \\ -x_2 + 2x_3 &\leq 3 \\ 3x_1 - 4x_2 - x_3 &\leq 5 \\ x_{1,2,3} &\geq 0 \\ x_{1,2,3} &\in \mathbb{Z} \end{aligned}$$

d)

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &\to \max \\ x_1 + x_2 + 2x_3 &\leq 4 \\ 2x_1 + x_3 &\leq 5 \\ 2x_1 + x_2 + 3x_3 &\leq 7 \\ x_{1,2,3} &\geq 0 \\ x_{1,2,3} &\in \mathbb{Z} \end{aligned}$$

6.2. Factory produces laptops and computers. It uses 1 000 kg Cu, 7 000 kg Al, 1 000 kg steel for its production 1 000 pieces of computers. It is necessary to expend 3 000 kg Cu, 1 000 kg Al, 1 000 kg Pb and 1 000 kg of steel in order to produce 1 000 pieces of laptops. The factory has available 6 000 kg Cu, 35 000 kg Al, 3 000 kg Pb and 7 000 kg steel. Maximize sales turnover when the computer price is 700 € and laptop price is 900 € per one piece.

6.3. We have 30 bar pieces each with the length of 10 meters. We need to cut 15 bar pieces with the length of 5 meters, 36 bar pieces with the length of 3 meters and 20 bar pieces with the length of 4 meters. Suggest an optimal solution by minimizing the scrap.

6.6 Solutions

6.1 a) $\boldsymbol{x}^{opt} = (40; 26)^\top$, $f^{opt} = 158$

b) $\boldsymbol{x}^{opt} = (2; 1)^\top$, $f^{opt} = 274$

c) The relaxation of ILPP is feasible unbounded, so ILPP is feasible unbounded or infeasible.

d) $\boldsymbol{x}^{opt} = (2; 0; 1)^\top$, $f^{opt} = 10$

6.2 The factory will gain maximum sales turnover if it produces only 5 000 pieces of computers. The sales turnover will be 3 500 000 €.

6.3 Optimal cutting of the bars: to cut 8 bars into $5+5$ meters and 20 bars into $4+3+3$ meters. The scrap will be zero.

Index

Lexicon – Vocabulary

English – Slovak

A

- Activity Analysis Problem	- úloha o plánovaní výroby
- Additivity	- aditivita
- Algorithm	- algoritmus
- Artificial LPP	- pomocná úloha LP
- Artificial Task	- pomocná úloha
- Artificial Variables	- pomocné premenné
- Assignment Problem	- priraďovací problém
- Auxiliary Tasks	- pomocná úloha LP

B

- Base of Vector Space	- báza vektorového priestoru
- Base	- báza
- Basic	- bázický
- Basic Solution	- bázické riešenie
- Basis	- báza
- Basic Feasible Solution	- bázické prípustné riešenie
- Bounded	- ohraničený

C

- Canonical Form	- kanonický tvar (úlohy LP)
- Change in objective function value	- zmena hodnoty účelovej funkcie
- Coefficients of Constraints	- koeficienty ohraničení
- Coefficients of Right Sides	- koeficienty pravých strán
- Complementary Slackness Theorem	- veta o komplementarite
- Constraints	- obmedzenia
- Convex Analysis	- konvexná analýza
- Convex Combination	- konvexná kombinácia
- Convex Hull	- konvexný obal

- Problem	- problém
- Proportionality	- proporcionalita
- Polygon	- mnohouholník
- Polyhedron	- mnohosten
- Primal Problem	- primárny problém
- Primal-Dual Pair	- primárno-duálna dvojica

Q

-	-

R

- Relaxation	- relaxácia
- Relaxation of ILP Problem	- relaxácia úlohy CLP
- Relative Price	- relatívna cena

S

- Segment Line	- úsečka
- Schedule Production Plan	- výrobný program
- Set of Feasible Solutions	- množina prípustných riešení
- Simplex Method	- simplexová metóda
- Simplex Table	- simplexová tabuľka
- Slack Variables	- doplnkové premenné
- Solution	- riešenie
- Standard Form	- štandardný tvar (úlohy LP)
- Strong Duality Theorem	- silná veta o dualite
- Submatrix	- podmatica

T

- Task	- úloha
- Transportation Problem	- dopravná úloha
- Two-Phase Algorithm of SM	- dvojfázový algoritmus pre SM

U

- Unbounded	- neohraničený
- Unbounded LPP	- neohraničená úloha LP

V

- Variables	- premenné

W

- Weak Duality Theorem	- slabá veta o dualite

Z

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Názov:	MATHEMATICS 3
Autor:	© RNDr. Štefan BEREŽNÝ, PhD., 2014
	© RNDr. Daniela KRAVECOVÁ, PhD., 2014
Recenzovali:	prof. RNDr. Marie Demlová, CSc.
	prof RNDr. Ján Plavka, CSc.
	prof RNDr. Michal Tkáč, CSc.
Vydavateľ:	Technická univerzita v Košiciach
	Fakulta elektrotechniky a informatiky
Miesto vydania:	Košice
Rok vydania:	2014
Vydanie:	Prvé
Rozsah:	122 strán
Náklad:	100 ks
ISBN:	978-80-553-1790-8